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MASTERS OF SCIENCE-MATHEMATICS

SEMESTER -II

ORDINARY DIFFERENTIAL EQUATION

DEMATH2SCORE3

BLOCK-1

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

ORDINARY DIFFERENTIAL EQUATION

BLOCK-I

BLOCK-II

BLOCK-1 ORDINARY DIFFERENTIAL EQUATION

Introduction To The Block-I

Unit 1 SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS : Deals with systems of Differential Equation and solution

Unit 2 FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS : Deals with first order ordinary Differential Equation with solution and properties

Unit 3 SECOND ORDER DIFFERENTIAL EQUATIONS : Deals with second order ordinary Differential Equation with solution and properties

Unit 4 POWER SERIES METOD WITH BESSEL FUNCTION : Deals with power series and solution of Bessel Function

Unit 5 EXISTENCE AND UNIQUENESS OF INITIAL VALUE PROBLEMS : Deals with existence and uniqueness if Initial Value problems with its examples

Unit 6 GRONWALL's INEQUALITY AND CONTINUATION OF SOLUTIONS : Deals with Gronwall's Inequality and continuation of solution with its examples

Unit 7 MAXIMAL INTERVAL OF EXISTENCE AND CONTINUOUS DEPENDENCE : Deals with Maximal interval of Existence and continuous dependence with its examples

UNIT 1 SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

STRUCTURE

1.0 OBJECTIVES

This chapter will be devoted to explaining the main concepts of the systems of linear differential equations. Some theorems concerning the fundamental matrix of such systems will be proved. Relations between Wronskian and linear independence/dependence of solutions of such systems will be developed.

1.1 INTRODUCTION

We have already studied single differential equation of different types and obtained the existence and uniqueness of solution of initial value problem of first order equations which are not necessarily linear. But in some practical situations we have to deal with more than one differential equation with many variables or depending upon a single variable. Such system of equations arise quite naturally in the analysis of certain physical situations. There is a very important class of differential equations known as linear differential equations, for which a general and elaborate theory is available. Apart from their theoretical importance, these equations are of great significance in physics and engineering in the problem of oscillation and electric circuits among others. This chapter extends the theory to a system of linear equations which give rise to the study of matrix differential equation, which will include both homogeneous and non-homogeneous type.

1.2 SYSTEMS OF DIFFERENTIAL EQUATIONS

In the introduction to this section we briefly discussed how a system of differential equations can arise from a population problem in which we keep track of the population of both the prey and the predator. It makes sense that the number of prey present will affect the number of the predator present. Likewise, the number of predator present will affect the number of prey present. Therefore the differential equation that governs the population of either the prey or the predator should in some way depend on the population of the other. This will lead to two differential equations that must be solved simultaneously in order to determine the population of the prey and the predator.

The whole point of this is to notice that systems of differential equations can arise quite easily from naturally occurring situations. Developing an effective predator-prey system of differential equations is not the subject of this chapter. However, systems can arise from nthnth order linear differential equations as well. Before we get into this however, let’s write down a system and get some terminology out of the way.

We are going to be looking at first order, linear systems of differential equations. These terms mean the same thing that they have meant up to this point. The largest derivative anywhere in the system will be a first derivative and all unknown functions and their derivatives will only occur to the first power and will not be multiplied by other unknown functions.

We call this kind of system a coupled system since knowledge of x2 is required in order to find x1x1 and likewise knowledge of x1 is required to find x2. We will worry about how to go about solving these later. At this point we are only interested in becoming familiar with some of the basics of systems.

Now, as mentioned earlier, we can write an nthnth order linear differential equation as a system. Let's see how that can be done.

1.2.1 Systems of Linear Differential Equations

A **system of linear differential equations** is a set of linear equations relating a group of functions to their derivatives. Because they involve functions and their derivatives, each of these linear equations is itself a differential equation. For example, $f'(x)=f(x)+g(x)$ is a linear equation relating 'f' to f and g, but $f'=fg$ is not, because the fg term is not linear. These equations can be solved by writing them in matrix form, and then working with them almost as if they were standard differential equations.

Systems of differential equations can be used to model a variety of physical systems, such as predator-prey interactions, but linear systems are the only systems that can be consistently solved explicitly.

Differential equations are models of real systems that are believed to change their states continuously, or, to put it more precisely, at infinitesimally short intervals in time. Differential equations, or rather systems of differential equations, connect a change in the state of a system to its current state, or even the change in a change of the state of the same system, in a way that is comparable to the way *difference equations* allow the calculation of future states of a system from its current state. But unlike difference equations, the application of differential equations supposes that the processes within the system modeled by these equations are continuous in time, whereas with difference equations, processes are discrete in time.

For a number of real systems, the approach of differential equations seems appropriate, for instance in the case of the movement of an arrow through the air or of the local concentration of some pollutant in a lake. Here, one is only (or at least mainly) interested in the current value of some continuously measurable variable that is seen as varying continuously over time. More generally speaking, *t* is the parameter of a process $\{xt,\ t\varepsilon\ T\}$ where *T* is a continuous set with the same cardinality as that of the set of real numbers; the general form of a (first-order ordinary) difference equation is

$$dx/dt = \dot{x} = f(x)$$

Here, in a more symbolic way, *dx* is the change that occurs to the state variable *x* of the system in question during the infinitesimally short time interval *dt* at any time *t*. Differential equations of higher order are also possible; a second-order differential equation has the general form

$$d^2x/dt^2 = \ddot{x} = f(x)$$

and is often transformed into a system of differential equations

$$\dot{y} = \ddot{x} = f(x) = g(y)$$

$$y = \dot{x} = h(x)$$

Strictly speaking, in the realm of the social and economic sciences, applications of differential equations and systems of them are only approximations, because the state variables of social and economic systems cannot undergo continuous changes. In demography, for example, we can only talk about the birth and death of an integer number of people, and in economics we can only calculate with a fixed number of products sold to the customer (not even with the exception of fluid, gaseous goods, or energy, which can be physically split down to molecules and energy quanta). In social psychology, it is still an open question whether attitudes change continuously (they are usually measured on four- or seven-point scales). And even if all these variables were continuous, the question remains whether these changes occur in a continuous manner: Children are born at a certain point in time, prices are paid at a certain point in time, and until the next payment arrives in one's bank account, the balance is constant.

On the other hand, with a large number of demographic events or financial transactions, one could argue that a differential equation is a sufficiently good approximation that is, in most cases, more easily treatable than the discrete event formalization of the real process (this even applies when the alternative is a deterministic difference equation). Differential equations can also treat probabilistic problems (then we have stochastic differential equations) and can describe processes in time and space, for instance in diffusion processes where the distribution of local concentrations or frequencies changes over time.

1.2.2 Unrestricted Growth

Linear differential equations of the type $\dot{x} = \lambda x$ and systems of such equations can always be solved, that is, it is always possible to write down the time-dependent function that obeys the differential equation (which is the exponential function $x(t) = Ae\,\lambda t$, where A and λ are two constants that depend on the initial condition and the proportionality constant between $\dot{x}$ and x, respectively). If the proportionality constant is positive, this results in an infinite growth, whereas with a negative proportionality ("the higher the value of x, the higher its decrease") the value of x approaches 0, though only in infinite time. This differential equation was first used in Thomas Malthus's (1766-1834) theories of demographic and economic growth.

A system of linear differential equations has a vector-valued exponential function as its solution. One of the earlier applications of a very simple system of linear differential equations was Lewis Fry Richardson's (1960) model of an arms race between two powers. The idea behind this model is that each block increases the armament budget both proportional to the current armament expenses of the other block and the budget available for other purposes. Thus, the change in the armament budget of block 1 is

$$\dot{x} = m\,(x_{max} - x) + ay = g + mx + ay \text{ with } g = mx_{max}$$

The same holds for the other block:

$$\dot{y} = bx + n\,(y_{max} - y) = h + bx + ny \text{ with } h = ny_{max}$$

The analytical solution for this system of two linear differential equations has the general form

$$q\,(t) = \theta_1 q_1\, exp\,(\lambda_1 t)\ \theta_2 q_2 \exp\,(\lambda_2 t) + q_3$$

where q and $q^{\bullet}$ are vectors with elements x and y and elements $x^{\bullet}$ and $y^{\bullet}$ respectively, while θ_1, θ_2, λ_1, λ_2, q_1, q_2 and q_3 are constants that depend on a, b, g, h, m, and n. In a way, only λ_1 and λ_2 are of special interest, because they are—as multipliers in the arguments to the exponential functions in the analytical solution—responsible for the overall behavior of the system. They can be shown to be the eigen-values of the matrix formed of $-m$, a, b, and $-n$, and these eigen-values can be complex, which means that besides stationary solutions, periodic solutions are also possible, at least in principle (although not in this case, where m, a, b, and n are all positive). If both λ_1 and λ_2 are negative, $q\,(t)$ approaches q_3 as times goes by; if at least one of them is positive, $q\,(t)$ grows beyond all limits (which of course is impossible in the real world).

1.2.3 Logistic Growth

One of the simplest cases of a differential equation in one variable—which also displays some interesting behavior— is the so-called logistic or Verhulst equation, which in its time-continuous version has the form

$$\dot{x} = rx\,(k - x)$$

One of the interpretations of this equation is that it describes a population in a habitat with carrying capacity K whose size changes continuously in such a way that the relative change ($x^{\bullet}/x$) is proportional both to the current size *x* and to the difference between the current size and the carrying capacity ($K - x$, this difference is the proportion of the habitat that, in a way, is so far unused).

The equation has two stationary solutions, namely, $x_{st\,0} = 0$ and $x_{st\,1} = K$. The former is unstable: Even from the tiniest initial state, the population will grow until the carrying capacity is exactly exhausted. The time-dependent function $x(t)$, which obeys the differential equation, is a monotonically growing function whose graph is an S- shaped curve. This time-dependent function can be written as

$$x(t) = Kx(0) \exp(rt)/\{K - x(0)\,[1 - \exp(rt)]\}$$

This differential equation is one of the simplest nonlinear ordinary differential equations.

1.2.4 The Lotks-Volterra Equation

Another well-known system of nonlinear differential equations is the so-called Lotka-Volterra equation, which describes the interaction between predators and prey. It can also be applied to the interaction between a human population (predator) and its natural resources (prey). Here, the relative growth of the prey is a sum of a (positive) constant and a negative term that is proportional to the size of the predator population, whereas the relative growth of the predator population is a sum of a (negative) constant and a positive term that is proportional to the size of the prey population. In other words, in the absence of the predator population the prey would grow infinitely, whereas in the absence of the prey, the predator population would die out.

$$\dot{x} = x(a - by)$$

$$\dot{y} = y(-c + dx)$$

This system of differential equations does not have a closed solution, but it has a number of interesting features that show up no matter how detailed the model is for the interaction between predators and prey: The solution for this system of differential equations is a periodic function with constant amplitude that depends on the initial condition. There is only one stationary state of the system, which is defined by $y = ab$ (this leads to $x^\bullet = 0$) and $x = c/d$ (this leads to $y^\bullet = 0$); thus if both hold, then no change will happen to the state of the system. Otherwise the populations increase and decrease periodically without ever dying out.

1.2.5 Partial Differential Equations

In most applications of differential equations and their systems, the parameter variable will be time, as in the examples above. But it is also possible to treat changes both in time and space with the help of a special type of differential equation, namely, *partial differential equations*. They define the change of the value of some attribute at some point in space and time—for instance, the expected change K of the continuously modeled and measured attitude X of a person that has the value x at time t, where this change will be different for different x and perhaps also for different t —in terms of this point in time and space. Thus,

$$K(x, t) = dx/dt = \partial V(x, t)/\partial x$$

For an application, see the next paragraph. Partial differential equations are seldom used in the social sciences because, typically, continuous properties of individual human beings—if they exist at all in the focus of interest of social scientists and economists—are difficult to measure, and even more difficult to measure within time intervals that are short enough to estimate any parameters of functions such as K and V in the above equation.

1.2.6 Stochastic Differential Equations

Stochastic influences can also be inserted into the formulation of differential equations. The simplest case is the so-called Langevin equation, which describes the motion of a system in its state space when there is both a potential whose gradient it follows and some stochastic influence that prevents the system from following the gradient of this potential in a precise manner. This type of description can, for instance, be used to describe the attitudes of voters during an election campaign.

Each voter's attitude can be defined (and measured) in a continuous attitude space. Their motions through this attitude space (say, from left to right; see, e.g., Downs 1957, p. 117) are determined by a "potential" that is determined either by some parties that "attract" voters toward their own positions in the same attitude space or by the "political climate" defined by the frequency distribution over the attitude space. In the latter case, voters would give up their attitude if it is shared by only a few and change it into an attitude that is more frequent. Thus they follow a gradient toward more frequent attitudes; but while moving through the attitude space, they would also perform random changes in their attitudes, thus not obeying exactly the overall political climate. And by changing individual attitudes, the overall "climate" or potential is changed. The movement could be described as follows:

$$\dot{q}(t) = -\gamma \partial V(q, t)/\partial q + \varepsilon_t$$

where

$$V(q, t) = -\text{In} f(q, t)$$

and $f(q, t)$ is the frequency distributions of voters over the attitude space at time t (V would be a polynomial up to some even order in q). One would typically find voters more or less normally distributed at the beginning of an election campaign, but the process described here would explain why and how polarization—a bimodal or multimodal frequency distribution—could occur toward the election date (Troitzsch 1990).

Check In Progress-I

Q. 1 Define System of Differential Equation.

Solution : . . . .

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Q. 2 Define Partial Differential Equation.

Solution : . . . .

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1.3 SOLVING SYSTEMS OF DIFFERENTIAL EQUATIONS

Imagine a distant part of the country where the life form is a type of cattle we'll call the 'xnay beast' that eats a certain type of grass we'll call 'ystrain grass'. The change in the xnay population depends on the ystrain as well as on the current size of the xnay population. The population of ystrain also depends on xnay and the current amount of ystrain. It's a fascinating mix! The more xnay, the more ystrain gets eaten which reduces the amount of ystrain which can reduce the amount of xnay. Less xnay and the ystrain can thrive. It can be very interesting! Especially for the xnay.

Having a variable whose rate of change depends on the variable itself, leads to exponential solutions of the differential equation. When we have two or more variables that are also interdependent, we have a **system of differential equations** and the solution is a mix of exponentials. Population problems are often modeled with systems of differential equations. In this lesson, we will look at two solution methods.

Describing the Equations

There are usually more than two interrelated variables in a population study. Food resources, predators, climate conditions, … will all interact with population size and its rate of change. To keep things simple, we will look at two variables. This will be enough to show the basic ideas of how to solve these systems of equations. For example:

$$\dot{x} = x - 3y$$

$$\dot{y} = 4x - 6y$$

with $x(0) = 2$ and $y(0) = 1$.

The little dot over the *x* and the *y* on the left-hand side, is the time derivative. The first equation says the time rate of change of *x* depends on both *x* and *y*. The same can be said for the time rate of change of *y*. It depends on both *x* and *y*. Solving these equations tells us how *x* and *y* evolve over time. The statement $x(0) = 2$ means at time $t = 0$, the population of xnay was 2. The population of ystrain was 1 at time $t = 0$.

We will use the eigenvalue and the Laplace transform methods to solve this system of equations. You are invited to check out other lessons on linear algebra and Laplace transforms for more details.

Solving Using Eigenvalues

We write the equations in matrix form:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & -3 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The matrix is called the 'A matrix'. In general, another term may be added to these equations. With no other term, the equations are called **homogeneous equations**. We will only look at the homogeneous case in this lesson.

The next step is to obtain the **characteristic equation** by computing the determinant of A - λI = 0. The details:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & -3 \\ 4 & -6-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(-6-\lambda) + 12$$

$$= \lambda^2 + 5\lambda + 6$$

$$= (\lambda + 3)(\lambda + 2) = 0$$

Notes

This tells us λ is -3 and -2. These are the eigenvalues of our system. Sometimes the eigenvalues are repeated and sometimes they are complex conjugate eigenvalues. In our example, we have two distinct and real eigenvalues. We will not cover the other cases in this lesson. Each of these eigenvalues has an eigenvector. For λ = -3, the eigenvector is calculated:

$$\begin{bmatrix} 1-(-3) & -3 \\ 4 & -6-(-3) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The equation relating *a* with *b* is 4*a* - 3*b* = 0. We choose a value for one of the letters. For example, letting *b* = 1 means *a* = ¾. The eigenvector v1 is

$$v_1 = \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix}$$

For λ = -2, the eigenvector calculation is:

$$\begin{bmatrix} 1-(-2) & -3 \\ 4 & -6-(-2) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The resulting equations are 3*a* - 3*b* = 0 and 4*a* - 4*b* = 0. These equations are true for *a* = *b*. Again, we choose a value. If *a* = 1, then *b* = 1. The eigenvector v2 is

$$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We now have a solution! In general, the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

where λ1 is -3 and λ2 is -2.

Substituting our eigenvalues and eigenvectors:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{3}{4} \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Linear Systems

Linear systems are systems of equations in which the variables are never multiplied with each other but only with constants and then summed up. Linear systems are used to describe both static and dynamic relations between variables.

In the case of the description of static relations, systems of linear algebraic equations describe invariants between variables such as:

$a_{11}x + a_{12}x_1 = c_1$

$a_{21}x + a_{22}x_2 = c_2$

Here, one would be interested in the values of x_1 and x_2 for which both equations hold. This system of equations can easily be written in matrix form:

or, more concisely:

$Ax = c$

The solution can be written as $x = A^{-1} c$ if the matrix A is invertible. Another frequent application of systems of linear algebraic equations is the following:

$y_i = b_0 + b_1x_{i\,1} + b_2x_{i\,2} + b_3x_{i\,3} + \ldots + b_mx_{i\,m} + e_i$

The above is a regression equation stating that for every object *i*, its attribute *Y* has a value that can be expressed as the weighted sum of its attributes X_1 through X_m, with a measurement error of E (whose variance should be a minimum in the classical regression analysis). The term linear is derived from the fact that the graphical representation of the above equation for $m = 1$ is a straight line (with intercept b_0 and slope b_1). The above equation holds for all i, thus the system of equations:

$$y_1 = b_0 + b_1 x_{11} + b_2 x_{12} + b_3 x_{13} + \ldots + b_m x_{1m} + e_1$$

$$y_2 = b_0 + b_1 x_{21} + b_2 x_{22} + b_3 x_{23} + \ldots + b_m x_{2m} + e_2$$

…

$$y_n = b_0 + b_1 x_{n1} + b_2 x_{n2} + b_3 x_{n3} + \ldots + b_m x_{nm} + e_n$$

for all n objects is often written in the abbreviated form, using matrices and vectors,

$$y = Xb + e$$

where y and e are column vectors containing all y_i and e_i, respectively ($i = 1 \ldots n$), b is a row vector containing all b_j ($j = 0 \ldots m$), and X is a $n \times (m + 1)$ matrix (with n rows and $m + 1$ columns) containing x_{ij} in the cells in row i and column j (where $x_{i0} = 1$ for all i). Here one is interested in the values of the regression coefficients b_j ($j = 0 \ldots m$) which minimize the variance of the regression residual E. This is solved by calculating $e^T e$ which is the sum of the squares of the still unknown e_i and calculating the derivatives of $e^T e$ with respect to all (also unknown) b_j. These derivatives will be 0 for $e^T e = min$, and the solution of this minimization problem is expressed as follows:

Linear systems are also used to describe dynamic relationships between variables. An early standard example from political science is English physicist Lewis Fry Richardson's (1881-1953) model of arms races, which consists of the following simplifying hypotheses:

- The higher the armament expenses of one military block, the faster the increase of the other block's armament expenses (as the latter wants to adapt to the threat as quickly as it can).

- The higher the armament expenses of a military block, the slower those expenses will increase (as it becomes more difficult to increase the proportion of military expenses with respect to the gross national product).

Calling the armament expenses of the two blocks x_1 and x_2, respectively, and their increase rates $x\dot{x}_1$ and $x\dot{x}_2$, respectively, one can model the increase rates as proportional both to the military expenses of the other block and the nonmilitary expenses of the own block (the total max max expenses and being constant), with some proportionality constants *a*, *b*, *m*, and *n* :

or, in shorter form:

Such linear systems of differential equations usually have a closed solution, that is, there is a vector-valued function $q(t)$ that fulfills this vector-valued differential equation. Usually, the precise form of $q(t)$ is not very interesting, but generally speaking it has the form:

$$q(t) = \theta_1 q_1 e^{\lambda 1 t} + \theta_2 q_2 e^{\lambda 2 t} + q_s$$

where q_1, q_2 and q_s are constant vectors and θ_1, θ_2 λ_1 and λ_2 are constant numbers of which mainly q_s, λ_1 and λ_2 are of special interest. q_s is the stationary state of the system of differential equations, that is, once the system has acquired this state, it will never leave that state, as the derivatives with respect to time vanish. λ_1 and λ_2 are the so-called eigenvalues of the matrix *A*, which, as the exponents of the two exponential functions in the above equation, determine whether the function $q(t)$ will grow beyond all limits over time or whether the two first terms in the right-hand side of the above equation will vanish as time approaches infinity. For negative values of both λ_1 and λ_2 the latter will happen, and the overall function will approach its stationary state (in which case the stationary state is called *stable* or an *attractor* or *sink*). If both eigenvalues are positive, then the function will grow beyond all limits (in which case the stationary state is called *unstable* or a *repellor* or *source*). If one of the λ_s is positive while the other is negative, the stationary is also unstable, but is called a *saddle* *point* because the function will first approach the stationary state and then move away. There is a special case when λ_1 and λ_2 are pure imaginary numbers—which happens if $4ab < 0$ and $m = n = 0$, as . In the arms race model this is not a reasonable assumption, as one block would increase its arms expenses faster, the smaller the arms expenses of the other block are (exactly one of a and b would be negative and the respective block would behave strangely, the other constant would be positive, so the other block would behave normally) and both would increase or decrease its arms expenses regardless of what their current values are ($m = n = 0$ means that there is no influence of the current value of arms expenses upon its change for both blocks). In this case both variables x_1 and x_2 oscillate around the stationary state.

The example demonstrates that systems of linear differential equations always have a closed solution, which can be expressed in several different forms. There is always exactly one stationary state (except in the case that the matrix A is not invertible) that can either be a sink or a source or a saddle or a center. Nonlinear systems often have more than one stationary state, but their behavior can be analyzed in a similar way, taking into account that a linear approximation of a nonlinear system behaves approximately the same in a small neighborhood of each of its stationary states.

1.4 ADAMS' METHOD

Adams' method is a numerical method for solving linear first-order ordinary differential equations of the form

$$\frac{dy}{dx} = f(x, y). \quad (1)$$

Let

$$h = x_{n+1} - x_n \quad (2)$$

be the step interval, and consider the Maclaurin series of y about x_n,

$$y_{n+1} = y_n + \left(\frac{d\,y}{d\,x}\right)_n (x - x_n) + \frac{1}{2}\left(\frac{d^2\,y}{d\,x^2}\right)_n (x - x_n)^2 + \ldots \tag{3}$$

$$\left(\frac{d\,y}{d\,x}\right)_{n+1} = \left(\frac{d\,y}{d\,x}\right)_n + \left(\frac{d^2\,y}{d\,x^2}\right)_n (x - x_n)^2 + \ldots. \tag{4}$$

Here, the derivatives of y are given by the backward differences

$$q_n \equiv \left(\frac{d\,y}{d\,x}\right)_n = \frac{\Delta\,y_n}{x_{n+1} - x_n} = \frac{y_{n+1} - y_n}{h} \tag{5}$$

$$\nabla q_n \equiv \left(\frac{d^2\,y}{d\,x^2}\right)_n = q_n - q_{n-1} \tag{6}$$

$$\nabla^2 q_n \equiv \left(\frac{d^3\,y}{d\,x^3}\right)_n = \nabla q_n - \nabla q_{n-1}, \tag{7}$$

etc. Note that by (◇), q_n is just the value of $f(x_n, y_n)$.

For first-order interpolation, the method proceeds by iterating the expression

$$y_{n+1} = y_n + q_n\,h \tag{8}$$

where $q_n \equiv f(x_n, y_n)$. The method can then be extended to arbitrary order using the finite difference integration formula from Beyer (1987)

$$\int_0^1 f_p\,d\,p = \left(1 + \tfrac{1}{2}\,\nabla + \tfrac{5}{12}\,\nabla^2 + \tfrac{3}{8}\,\nabla^3 + \tfrac{251}{720}\,\nabla^4 + \tfrac{95}{288}\,\nabla^5 + \tfrac{19087}{60480}\,\nabla^6 + \ldots\right) f \tag{9}$$

to obtain

$$y_{n+1} - y_n = h\left(q_n + \tfrac{1}{2}\,\nabla q_{n-1} + \tfrac{5}{12}\,\nabla^2 q_{n-2} + \tfrac{3}{8}\,\nabla^3 q_{n-3} + \tfrac{251}{720}\,\nabla^4 q_{n-4} + \tfrac{95}{288}\,\nabla^5 q_{n-5} + \ldots\right). \tag{10}$$

Note that von Kármán and Biot (1940) confusingly use the symbol normally used for forward differences δ to denote backward differences ∇.

Check In Progress-II

Notes

Q. 1 Define Linear Systems .

Solution : . . . .
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Q. 2 State Adam's Method .

Solution : . . . .
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1.5 LINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION AND SYSTEM

A partial differential equation (or system) of the form

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f$$

for which at any point $x = (x_0, \dots, x_n)$ of its domain of definition Ω one can distinguish among the real variables $y_0, \dots, y_n$ (if necessary, after a suitable affine transformation of the independent variables) one variable, say $y_0 = \lambda$, in a such a way that for all points $Y = (y_1, \dots, y_n)$ of the n-dimensional Euclidean space $\mathbf{R}^n$ the characteristic equation (in λ)

$$\det \sum_{|\alpha| \leq m} a_\alpha(x) y^\alpha \Big|_{y_0 = \lambda}, \qquad y^\alpha = (y_0^{\alpha_0}, \dots, y_n^{\alpha_n}),$$

has exactly Nm real roots. Here $\alpha = (\alpha_0, \dots, \alpha_n)$ is a vector with non-negative integer coordinates, $|\alpha| = \sum_{j=0}^{n} \alpha_j$ is the order of the differential operator $D^{\alpha} = D^{\alpha_0} \dots D_n^{\alpha_n}$, $D_j = \partial / \partial x_j$, $j = 0, \dots, n$, m is the order of the system (1), $a_{\alpha}(x)$ is a real square matrix of order N, defined in Ω, $u(x) = \|u_j(x)\|$, $j = 1, \dots, N$, is an unknown column vector, and $f(x)$ is a vector with N components, defined in Ω.

A typical example is the wave equation

$$u_{x_0 x_0} = \sum_{j=1}^{n} u_{x_j x_j} \ .$$

Many problems in mathematical physics reduce to linear hyperbolic partial differential equations or systems of equations.

A subset $S\colon \phi(x) = 0$ is said to be characteristic at a point x if $\mathrm{grad}\ \phi \neq 0$ and $Q(x, \mathrm{grad}\ \phi) = 0$, where

$$Q(x, y) \equiv Q(x, y_0, Y) = \det \sum_{|\alpha| = m} a_{\alpha}(x) y^{\alpha}$$

is the characteristic form of the system (1). If $Q(x, y) = 0$, then one says that the vector y defines a characteristic direction or characteristic normal at the point x. The surface S is called a characteristic surface (or characteristic) of the system (1) if

$$Q(x, \mathrm{grad}\ \phi) = 0 \qquad \text{for all } x \in S.$$

A surface which does not have characteristic normals at any point is called a free surface. On a free surface the rank of the characteristic matrix

$$A(x, y) \equiv A(x, y_0, Y) = \sum_{|\alpha| = m} a_{\alpha}(x) y^{\alpha}, \qquad y = \mathrm{grad}\ \phi,$$

is equal to N, while on a characteristic surface S it is less than N. A characteristic S is said to be simple if for some j and all $x \in S$,

$$\left(\frac{\partial}{\partial x_j}\right) A(x, \mathrm{grad}\ \phi) \neq 0 \ .$$

Otherwise the characteristic is said to be multiple. A characteristic is sometimes said to be simple if the rank of the matrix $A(x, \mathrm{grad}\ \phi)$ is $N - 1$.

The system (1) is said to be hyperbolic at the point x with respect to the hyperplane S_0: $x_0 = 0$ if the matrix $A(x, 1, Y) = a_{m,0,...,0}$ is non-singular (that is, the surface S is free) and if all roots $\lambda = \lambda_k$, $k = 1,..., mN$, of the characteristic equation $Q(x, \lambda, Y) = 0$ are real for all points $Y \in \mathbf{R}^n$. The system (1) is said to be hyperbolic in the domain Ω with respect to S_0 if it is hyperbolic with respect to S_0 at every point $x \in \Omega$.

An important class of hyperbolic equations and systems consists of strictly hyperbolic equations and systems, which are sometimes called fully hyperbolic systems, or systems, hyperbolic in the narrow sense. A system (1) is called a strictly hyperbolic system if all roots $\lambda = \lambda_k$ of the characteristic equation are distinct for any non-zero vector $Y \in \mathbf{R}^n$. The characteristics of a strictly hyperbolic equation (or system) are simple. Strictly hyperbolic (with respect to S_0) systems are notable for the fact that the Cauchy problem

$$D_0^k u|_{x_0=0} = \psi_k(x), \qquad k = 0,..., m-1,$$

for them is well-posed under the single assumption of sufficient smoothness of the coefficients $a_\alpha(x)$, $f(x)$ of the system (1) and of the initial data (initial functions) $\psi_k(x)$, $x = (x_0,..., x_n)$. There are examples of hyperbolic but not strictly hyperbolic equations of the form (1) (even with constant coefficients in front of the derivatives of order m) for which the Cauchy problem is ill-posed.

The solution $u(x)$ of the Cauchy problem (4) for the wave equation (3) can be written out explicitly, and when $m \equiv 0 (\bmod 2)$, and only then, it has the property that the value of $u(x)$ at the vertex x^* of the characteristic cone $|X - X^*| = x_0^* - x_0$, $X = (x_1,..., x_n)$, depends only on the value of the initial data $\psi_0(x)$ and $\psi_1(x)$ and their derivatives on the base $|X - X^*| = x_0^*$, $x_0 = 0$, of this cone (the so-called Huygens principle).

For strictly hyperbolic (with respect to S_0) equations and systems the question of the diffusion of waves and the related question of gaps have been investigated (cf. Lacuna). An exhaustive answer has been given in the case of an equation with constant coefficients of the form

$$\sum_{|\alpha|=m} a_\alpha D^\alpha u=0, \qquad a_\alpha = \text{ const}.$$

For the case of one (scalar) equation with constant coefficients $a_\alpha(x)=a_\alpha=\text{const}$ the definition of strict hyperbolicity has been generalized as follows. Equation (1) is said to be hyperbolic with respect to a non-zero vector $y\in\mathbf{R}^{n+1}$ if

$$\sum_{|\alpha|=m} a_\alpha y^\alpha \neq 0$$

and if there is a real number λ_0 such that for all $\xi=(\xi_0,\dots,\xi_n)\in\mathbf{R}^{n+1}$ and $\lambda>\lambda_0$,

$$\sum_{|\alpha|\leq m} a_\alpha (\lambda y+i\xi)^\alpha \neq 0.$$

Of all linear equations with constant coefficients, only for equations that are hyperbolic in this sense the Cauchy problem is well-posed for arbitrary sufficiently smooth initial functions defined on the hyperplane

$$\sum_{j=0}^{n} x_j y_j = 0.$$

In particular, the wave equation (3) is hyperbolic in this sense with respect to any vector for which

$$y_0^2 > \sum_{j=1}^{n} y_j^2.$$

There are various generalizations of the definition of strict hyperbolicity of equations and systems. These are mainly equations and systems that are completely characterized by the fact that the Cauchy problem with data on a free surface is uniquely solvable for them for any sufficiently smooth initial functions, without any restrictions on the growth at infinity.

Notes

Another important class of linear hyperbolic systems of the first order is the class of symmetric hyperbolic systems. The system

$$\sum_{j=0}^{n} a_j(x)u_{x_j} + b(x)u = f(x),$$

where $a_j(x)$, $b(x)$ are square matrices of order N, defined in Ω, and $u(x)$ is an unknown vector of N components, is called a symmetric hyperbolic system in Ω if the matrices $a_j(x)$ are symmetric (or are symmetrizable simultaneously by the same transformation) and if at every point there is a spatially-oriented hyperplane (or, space-like hyperplane), that is, a hyperplane whose normal $y = (y_0, \dots, y_n)$ has the property that the matrix $\sum_{j=0}^{n} y_j a_j(x)$ is positive definite. If for a symmetric hyperbolic system (5) with sufficiently smooth coefficients the given initial functions and the right-hand side have square-integrable generalized partial derivatives of order p, then there is a unique generalized solution of the Cauchy problem with the same number of square-integrable partial derivatives. Any strictly hyperbolic partial differential equation of the second order reduces to a symmetric hyperbolic system.

An equation (1) of the second order in the class of solutions regular in a domain Ω can be written in the form

$$\sum_{k,j=0}^{n} a_{kj}(x)u_{x_k x_j} + \sum_{j=0}^{n} b_j(x)u_{x_j} + c(x)u = f(x),$$

where $a_{kj}(x) = a_{jk}(x)$, $b_j(x)$, $c(x)$, and $f(x)$ are functions defined in Ω. Equation (6) is hyperbolic in Ω if at every point of Ω all eigen values of the matrix of leading coefficients $a(x) = \|a_{jk}(x)\|$, $k, j = 0, \dots, n$, are non-zero, and one of these eigen values differs in sign from all others. With respect to (6), along with the characteristic surface one can distinguish two types of smooth surfaces: spatially-oriented surfaces and time-oriented surfaces (also called space-like and time-like surfaces). If the surfaces are given by an equation of the form $\phi(x) = 0$, then on a surface of the first type $Q(x, \text{grad } \phi) > 0$, while on a surface of the second type $Q(x, \text{grad } \phi) < 0$, where

$$Q(x, y) = \sum_{k,j=0}^{n} a_{kj}(x) y_k y_j .$$

The Cauchy problem for hyperbolic partial differential equations with initial data on a time-like surface is generally not well-posed.

A hyperbolic partial differential equation

$$u_{x_0 x_0} - \sum_{k,j=1}^{n} a_{kj}(x) u_{x_k x_j} + \sum_{j=0}^{n} b_j(x) u_{x_j} + c(x) u = f(x)$$

is said to be uniformly (or regularly) hyperbolic in a domain Ω if there is a positive number ϵ such that

$$\sum_{k,j=1}^{n} a_{kj}(x)\, \xi_k \;\xi_j > \epsilon \sum_{j=1}^{n} \xi_j^2$$

for all x in $\overline{\Omega}$ and for any non-zero vector $(\xi_1 ,\dots, \xi_n) \in \mathbf{R}^n$. For $n = 1$ the inequality

$$a_{01}^2 - a_{00} a_{11} < 0 \quad \text{for all } x \in \overline{\Omega}$$

is a necessary and sufficient condition for (6) to be uniformly hyperbolic in Ω. The equation for the vibration of a string,

$$u_{x_0 x_0} - u_{x_1 x_1} = 0,$$

is a typical representative of a linear uniformly hyperbolic partial differential equation of the second order with two independent variables. The general solution of this equation in any convex domain Ω of the plane $\mathbf{R}^2$ is given by the d'Alembert formula:

$$u(x) = f(x_0 + x_1) + g(x_0 - x_1),$$

where f and g are arbitrary functions.

After a non-singular real change of variables x_0 and x_1, the hyperbolic partial differential equation (6) with $n = 1$ reduces to the normal (canonical) form

$$u_{y_0 y_0} - u_{y_1 y_1} + A(y) u_{y_0} + B(y) u_{y_1} + C(y) u = F(y),$$

$y = (y_0, y_1)$.

For hyperbolic systems written in the form (7), where $A(y)$, $B(y)$ and $C(y)$ are given real square matrices of order N, $F(y)$ is a given vector and u an unknown vector, both with N components, the question of the well-posedness of the Cauchy problem with initial data on a non-characteristic (free) curve and the Goursat problem with data on two intersecting characteristics have been completely investigated by the Riemann method.

The main problems about hyperbolic equations are the following: the Cauchy problem, the Cauchy characteristic problem and the mixed problem (see also Mixed and boundary value problems for hyperbolic equations and systems).

In the investigation of the main problems an important role is played by fundamental solutions, which make it possible to obtain explicit (integral) representations of regular and generalized solutions and to establish their structural and qualitative properties, in particular to study the question of wave fronts and the propagation of discontinuities.

Equation (6) is called an ultra-hyperbolic equation in a domain Ω if at every point $x \in \Omega$ all eigen values of the matrix $a(x)$ are non-zero and at least two of them differ in sign from all the others, of which there are at least two. An example of an ultra-hyperbolic equation is an equation of the form

$$\sum_{j=0}^{n} (u_{x_j x_j} - u_{y_j y_j}) = 0, \qquad n \geq 1,$$

which has the following property: If $u(x, y)$ is a regular solution of (8) in a domain Ω of the Euclidean space of the variables $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n)$ and if (x^*, y^*) is an arbitrary point of Ω, then the mean value of the function $u(x, y^*)$ calculated on the sphere $\sum_{j=0}^{n} (x_j - x_j^*)^2 = r^2$ with centre at the

point $x^* = (x_0^*, \dots, x_n^*)$ and radius r, is equal to the mean value of the function $u(x^*, y)$ calculated on the sphere $\sum_{j=0}^{n} (y_j - y_j^*)^2 = r^2$ with centre at the point $y^* = (y_0^*, \ldots, y_n^*)$ and the same radius r. This theorem is extensively used in the theory of linear hyperbolic partial differential equations of the second order with constant coefficients.

1.6 SUMMARY

- A **system of linear differential equations** is a set of linear equations relating a group of functions to their derivatives. Because they involve functions and their derivatives, each of these linear equations is itself a differential equation. For example, *f′*(*x*)=*f*(*x*)+*g*(*x*) is a linear equation relating 'f' to *f* and *g*, but *f′*=*fg* is not, because the *fg* term is not linear. These equations can be solved by writing them in matrix form, and then working with them almost as if they were standard differential equations.

- Describing Equations : There are usually more than two interrelated variables in a population study. Food resources, predators, climate conditions, … will all interact with population size and its rate of change. To keep things simple, we will look at two variables.

- A partial differential equation (or system) of the form

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f$$

for which at any point $x = (x_0, \ldots, x_n)$ of its domain of definition Ω one can distinguish among the real variables $y_0, \ldots, y_n$ (if necessary, after a suitable affine transformation of the independent variables) one variable

1.7 KEYWORD

Partial Differential : *Partial derivatives* are *defined* as *derivatives* of a function of multiple variables when all but the variable of interest are held fixed during the *differentiation*. ... A *differential* equation expressing one or more quantities in terms of *partial derivatives* is called a *partial differential* equation.

Lotks-Volterra : The Lotka–Volterra equations, also known as the predator–prey equations, are a pair of first-order nonlinear differential equations, frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey.

Linear Hyperbolic : Linear hyperbolic partial differential equation and system. ... The system (1) is said to be hyperbolic at the point with respect to the hyperplane : if the matrix is non-singular (that is, the surface is free) and if all roots , , of the characteristic equation are real for all points

1.8 EXERCISE

Q. 1 Define systems of linear differential Equation.

Q. 2 State and prove Adam's Equation.

Q. 3 Define systems of Partial Differential Equation.

Q. 4 Define system of linear hyperbolic partial differential equation.

Q. 5 Define Stochastic differential equation.

1.9 ANSWER TO CHECK IN PROGRESS

Check In Progress-1

Answer Q. 1 Check in section 3

Q. 2 Check in section 3.5

Check In progress-II

Answer Q. 1 Check in section 5

Q. 2 Check in section 6

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UNIT 2FIRST ODER ORDINARY DIFFERENTIAL EQUATIONS

STRUCTURE

2.0 OBJECTIVE

- Learn first order differential equation
- Learn differential equation
- Learn Homogeneous Differential Equaion
- Learn Solution of differential Equation

2.1 INTRODUCTION

In mathematics, an ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and the derivatives of those functions. The term *ordinary* is used in contrast with the term partial differential equation which may be with respect to *more than* one independent variable.

2.2 DIFFERENTIAL EQUATIONS

- A linear differential equation is a differential equation that is defined by a linear polynomial in the unknown function and its derivatives, that is an equation of the form

  a_o(x)y + a_1(x) y'

  e , ..., and are arbitrary differentiable functions that do not need to be linear, and are the successive derivatives of the unknown function *y* of the variable *x*.
- Among ordinary differential equations, linear differential equations play a prominent role for several reasons.
  Most elementary and special functions that are encountered in physics and applied mathematics are solutions of linear differential equations (see Holonomic function). When physical phenomena are modeled with non-linear equations, they are generally approximated by linear differential equations for an easier solution. The few non-linear ODEs that can be solved explicitly are generally solved by transforming the equation into an equivalent linear ODE (see, for example Riccati equation).
- Some ODEs can be solved explicitly in terms of known functions and integrals. When that is not possible, the equation for computing the Taylor series of the solutions may be useful. For applied problems, numerical methods for ordinary differential equations can supply an approximation of the solution.

2.2.1 First-Order Ordinary Differential Equation

Given a first-order ordinary differential equation

$$\frac{dy}{dx} = F(x, y), \qquad (1)$$

if $F(x, y)$ can be expressed using separation of variables as

Notes

$$F(x, y) = X(x)\, Y(y), \tag{2}$$

then the equation can be expressed as

$$\frac{d\,y}{Y(y)} = X(x)\, d\,x \tag{3}$$

and the equation can be solved by integrating both sides to obtain

$$\int \frac{d\,y}{Y(y)} = \int X(x)\, d\,x. \tag{4}$$

Any first-order ODE of the form

$$\frac{d\,y}{d\,x} + p(x)\, y = q(x) \tag{5}$$

can be solved by finding an integrating factor $\mu = \mu(x)$ such that

$$\frac{d}{d\,x}(\mu\, y) = \mu\,\frac{d\,y}{d\,x} + y\,\frac{d\,\mu}{d\,x} \tag{6}$$

$$= \mu\, q(x). \tag{7}$$

Dividing through by $\mu\, y$ yields

$$\frac{1}{y}\,\frac{d\,y}{d\,x} + \frac{1}{\mu}\,\frac{d\,\mu}{d\,x} = \frac{q(x)}{y}. \tag{8}$$

However, this condition enables us to explicitly determine the appropriate μ for arbitrary p and q. To accomplish this, take

$$p(x) = \frac{1}{\mu}\,\frac{d\,\mu}{d\,x} \tag{9}$$

in the above equation, from which we recover the original equation (◇), as required, in the form

$$\frac{1}{y}\,\frac{d\,y}{d\,x} + p(x) = \frac{q(x)}{y}. \tag{10}$$

But we can integrate both sides of (9) to obtain

$$\int p(x)\, d\,x = \int \frac{d\,\mu}{\mu} = \ln \mu + c \tag{11}$$

$$\mu = e^{\int p(x)\, d\,x}. \tag{12}$$

Now integrating both sides of (◇) gives

$$\mu y = \int \mu q(x)\, dx + c \tag{13}$$

(with μ now a known function), which can be solved for y to obtain

$$y = \frac{\int \mu q(x)\, dx + c}{\mu} = \frac{\int e^{\int^x p(x')\, dx'} q(x)\, dx + c}{e^{\int^x p(x')\, dx'}}, \tag{14}$$

where c is an arbitrary constant of integration.

Given an nth-order linear ODE with constant coefficients

$$\frac{d^n y}{d x^n} + a_{n-1} \frac{d^{n-1} y}{d x^{n-1}} + \ldots + a_1 \frac{d y}{d x} + a_0 y = Q(x), \tag{15}$$

first solve the characteristic equation obtained by writing

$$y \equiv e^{r x} \tag{16}$$

and setting $Q(x) = 0$ to obtain the n complex roots.

$$r^n e^{r x} + a_{n-1} r^{n-1} e^{r x} + \ldots + a_1 r e^{r x} + a_0 e^{r x} = 0 \tag{17}$$

$$r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0. \tag{18}$$

Factoring gives the roots r_i,

$$(r - r_1)(r - r_2) \cdots (r - r_n) = 0. \tag{19}$$

For a nonrepeated real root r, the corresponding solution is

$$y = e^{r x}. \tag{20}$$

If a real root r is repeated k times, the solutions are degenerate and the linearly independent solutions are

$$y = e^{r x},\ y = x e^{r x},\ \ldots,\ y = x^{k-1} e^{r x}. \tag{21}$$

Complex roots always come in complex conjugate pairs, $r_\pm = a \pm i b$. For nonrepeated complex roots, the solutions are

Notes

$$y = e^{ax}\cos(bx),\ y = e^{ax}\sin(bx). \tag{22}$$

If the complex roots are repeated k times, the linearly independent solutions are

$$y = e^{ax}\cos(bx),\ y = e^{ax}\sin(bx),\ \ldots,\ y = x^{k-1}e^{ax}\cos(bx),\ y = x^{k-1}e^{ax}\sin(\tag{23}$$

Linearly combining solutions of the appropriate types with arbitrary multiplicative constants then gives the complete solution. If initial conditions are specified, the constants can be explicitly determined. For example, consider the sixth-order linear ODE

$$(\tilde{D} - 1)(\tilde{D} - 2)^3\left(\tilde{D}^2 + \tilde{D} + 1\right)y = 0,$$

which has the characteristic equation

$$(r - 1)(r - 2)^3\left(r^2 + r + 1\right) = 0.$$

The roots are 1, 2 (three times), and $\left(-1 \pm \sqrt{3}\ i\right)/2$, so the solution is

$$y = A e^x + B e^{2x} + C x e^{2x} + D x^2 e^{2x} + E e^{-x/2}\cos\left(\tfrac{1}{2}\sqrt{3}\ x\right) + F e^{-x/2}\sin\left(\tfrac{1}{2}\sqrt{3}\ x\right.$$

If the original equation is nonhomogeneous ($Q(x) \neq 0$), now find the particular solution y^* by the method of variation of parameters. The general solution is then

$$y(x) = \sum_{i=1}^{n} c_i\, y_i(x) + y^*(x), \tag{27}$$

where the solutions to the linear equations are $y_1(x)$, $y_2(x)$, ..., $y_n(x)$, and $y^*(x)$ is the particular solution.

Check In Progress-I

Q. 1 Define first-order ordinary differential equation.

Solution : . . . .

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Q. 2 Define Differential Equation.

Solution : . . . .

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2.3 LINEAR ORDINARY DIFFERENTIAL EQUATION

An ordinary differential equation (cf. Differential equation, ordinary) that is linear in the unknown function of one independent variable and its derivatives, that is, an equation of the form

$$x^{(n)} + a_1(t)x^{(n-1)} + \dots + a_n(t)x = f(t),$$

where $x(t)$ is the unknown function and $a_i(t)$, $f(t)$ are given functions; the number n is called the order of equation (1) (below the general theory of linear ordinary differential equations is presented; for equations of the second order see also Linear ordinary differential equation of the second order).

1) If in (1) the functions $a_1, \dots, a_n, f$ are continuous on the interval (a, b), then for any numbers $x_0, x_0', \dots, x_0^{(n-1)}$ and $t_0 \in (a, b)$ there is a unique solution $x(t)$ of (1) defined on the whole interval (a, b) and satisfying the initial conditions

$$x(t_0) = x_0, x'(t_0) = x_0', \dots, x^{(n-1)}(t_0) = x_0^{(n-1)}.$$

The equation

Notes

$$x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = 0$$

is called the homogeneous equation corresponding to the inhomogeneous equation (1). If $x(t)$ is a solution of (2) and

$$x(t_0) = x'(t_0) = \ldots = x^{(n-1)}(t_0) = 0,$$

then $x(t) \equiv 0$. If $x_1(t)$,..., $x_m(t)$ are solutions of (2), then any linear combination

$$C_1 x_1(t) + \ldots + C_m x_m(t)$$

is a solution of (2). If the n functions

$$x_1(t) ,..., x_n(t)$$

are linearly independent solutions of (2), then for every solution $x(t)$ of (2) there are constants C_1 ,..., C_n such that

$$x(t) = C_1 x_1(t) + \ldots + C_n x_n(t) .$$

Thus, if (3) is a fundamental system of solutions of (2) (i.e. a system of n linearly independent solutions of (2)), then its general solution is given by (4), where C_1 ,..., C_n are arbitrary constants. For every non-singular $n \times n$ matrix $B = \|b_{ij}\|$ and every $t_0 \in (a, b)$ there is a fundamental system of solutions (3) of equation (2) such that

$$x_i^{(n-j)}(t_0) = b_{ij}, \qquad i, j = 1 ,..., n .$$

For the functions (3) the determinant

$$W(t) = \det \begin{Vmatrix} x_1(t) & \ldots & x_n(t) \\ x_1' & \ldots & x_n'(t) \\ \ldots & \ldots & \ldots \\ x_1^{(n-1)}(t) & \ldots & x_n^{(n-1)}(t) \end{Vmatrix}$$

is called the Wronski determinant, or Wronskian. If (3) is a fundamental system of solutions of (2), then $W(t) \neq 0$ for all $t \in (a, b)$.
If $W(t_0) = 0$ for at least one point t_0, then $W(t) \equiv 0$ and the solutions (3) of equation (2) are linearly dependent in this case. For the Wronskian

of the solutions (3) of equation (2) the Liouville–Ostrogradski formula holds:

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} a_1(\tau)\, d\tau\right).$$

The general solution of (1) is the sum of the general solution of the homogeneous equation (2) and a particular solution $x_0(t)$ of the inhomogeneous equation (1), and is given by the formula

$$x(t) = C_1 x_1(t) + \ldots + C_n x_n(t) + x_0(t),$$

where $x_1(t), \ldots, x_n(t)$ is a fundamental system of solutions of (2) and $C_1, \ldots, C_n$ are arbitrary constants. If a fundamental system of solutions (3) of equation (2) is known, then a particular solution of the inhomogeneous equation (1) can be found by the method of variation of constants.

2) A system of linear ordinary differential equations of order n is a system

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t) x_i + b_i(t), \qquad i = 1, \ldots, n,$$

or, in vector form,

$$\dot{x} = A(t)x + b(t),$$

where $x(t) \in \mathbf{R}^n$ is an unknown column vector, $A(t)$ is a square matrix of order n and $b(t)$ is a given vector function. Suppose also that $A(t)$ and $b(t)$ are continuous on some interval (a, b). In this case, for any $t_0 \in (a, b)$ and $x_0 \in \mathbf{R}^n$ there is a unique solution $x(t)$ of the system (5) defined on the whole interval (a, b) and satisfying the initial condition $x(t_0) = x_0$.

The linear system

$$\dot{x} = A(t)x$$

Notes

is called the homogeneous system corresponding to the inhomogeneous system (5). If $x(t)$ is a solution of (6) and $x(t_0)=0$, then $x(t)\equiv 0$; if $x_1(t)$,..., $x_m(t)$ are solutions, then any linear combination

$$C_1 x_1(t)+\ldots+C_m x_m(t)$$

is a solution of (6); if $x_1(t)$,..., $x_m(t)$ are linearly independent solutions of (6), then the vectors $x_1(t)$,..., $x_m(t)$ are linearly independent for any $t\in(a,b)$. If the n vector functions

$$x_1(t) ,..., x_n(t)$$

form a fundamental system of solutions of (6), then for every solution $x(t)$ of (6) there are constants C_1 ,..., C_n such that

$$x(t)=C_1 x_1(t)+\ldots+C_n x_n(t) .$$

Thus, formula (8) gives the general solution of (6). For any $t_0\in(a,b)$ and any linearly independent vectors a_1 ,..., $a_n\in\mathbf{R}^n$ there is a fundamental system of solutions (7) of the system (6) such that

$$x_1(t_0)=a_1 ,..., x_n(t_0)=a_n .$$

For vector functions (7) that are solutions of (6), the determinant $W(t)$ of the matrix

$$X(t)=\begin{Vmatrix} x_{11}(t) & \ldots & x_{n1}(t) \\ \cdots & \cdots & \cdots \\ x_{1n}(t) & \ldots & x_{nn}(t) \end{Vmatrix},$$

where $x_{ij}(t)$ is the j-th component of the i-th solution, is called the Wronski determinant, or Wronskian. If (7) is a fundamental system of solutions of (6), then $W(t)\neq 0$ for all $t\in(a,b)$ and (9) is called a fundamental matrix. If the solutions (7) of the system (6) are linearly dependent for at least one point t_0, then they are linearly dependent for any $t\in(a,b)$, and in this case $W(t)\equiv 0$. For the Wronskian of the solutions (7) of the system (6) Liouville's formula holds:

$$W = W(t_0)\exp\left(\int_{t_0}^{t} \mathrm{Tr}\,(A(\tau))\,d\tau\right),$$

where $\mathrm{Tr}\,(A(\tau)) = a_{11}(\tau) + \ldots + a_{nn}(\tau)$ is the trace of the matrix $A(\tau)$. The matrix (9) satisfies the matrix equation $\dot{X} = A(t)X(t)$. If $X(t)$ is a fundamental matrix of the system (6), then for every other fundamental matrix $Y(t)$ of this system there is a constant non-singular $n \times n$ matrix C such that $Y(t) = X(t)C$. If $X(t_0) = E$, where E is the unit matrix, then the fundamental matrix $X(t)$ is said to be normalized at the point t_0 and the formula $x(t) = X(t)x_0$ gives the solution of (6) satisfying the initial condition $x(t_0) = x_0$.

If the matrix $A(t)$ commutes with its integral, then the fundamental matrix of (6) normalized at the point $t_0 \in (a, b)$ is given by the formula

$$X(t) = \exp\left(\int_{t_0}^{t} A(\tau)\,d\tau\right).$$

In particular, for a constant matrix A the fundamental matrix normalized at the point t_0 is given by the formula $X(t) = \exp A(t - t_0)$. The general solution of (5) is the sum of the general solution of the homogeneous system (6) and a particular solution $x_0(t)$ of (5) and is given by the formula

$$x(t) = C_1 x_1(t) + \ldots + C_n x_n(t) + x_0(t),$$

where $x_1(t), \ldots, x_n(t)$ is a fundamental system of solutions of (6) and $C_1, \ldots, C_n$ are arbitrary constants. If a fundamental system of solutions (7) of the system (6) is known, then a particular solution of the inhomogeneous system (5) can be found by the method of variation of constants. If $X(t)$ is a fundamental matrix of the system (6), then the formula

$$x(t) = X(t)X^{-1}(t_0)x_0 + \int_{t_0}^{t} X(t)X^{-1}(\tau)b(\tau)\,d\tau$$

gives the solution of (5) satisfying the initial condition $x(t_0) = x_0$.

3) Suppose that in the system (5) and (6) $A(t)$ and $b(t)$ are continuous on a half-line $[\alpha, +\infty)$. All solutions of (5) are simultaneously either stable or unstable, so the system (5) is said to be stable (uniformly stable, asymptotically stable) if all its solutions are stable (respectively, uniformly stable, asymptotically stable, cf. Asymptotically-stable solution; Lyapunov stability). The system (5) is stable (uniformly stable, asymptotically stable) if and only if the system (6) is stable (respectively, uniformly stable, asymptotically stable). Therefore, in the investigation of questions on the stability of linear differential systems it suffices to consider only homogeneous systems.

The system (6) is stable if and only if all its solutions are bounded on the half-line $[\alpha, +\infty)$. The system (6) is asymptotically stable if and only if

$$\lim_{t \to +\infty} x(t) = 0$$

for all its solutions $x(t)$. The latter condition is equivalent to (10) being satisfied for n solutions $x_1(t), \dots, x_n(t)$ of the system that form a fundamental system of solutions. An asymptotically-stable system (6) is asymptotically stable in the large.

A linear system with constant coefficients

$$\dot{x} = Ax$$

is stable if and only if all eigen values $\lambda_1, \dots, \lambda_n$ of A have non-positive real parts (that is, $\operatorname{Re} \lambda_i \leq 0$, $i = 1, \dots, n$), and the eigen values with zero real part may have only simple elementary divisors. The system (11) is asymptotically stable if and only if all eigen values of A have negative real parts.

4) The system

$$\dot{y} = -A^T(t)y,$$

where $A^T(t)$ is the transposed matrix of $A(t)$, is called the adjoint system of the system (6). If $x(t)$ and $y(t)$ are arbitrary solutions of (6) and (12), respectively, then the scalar product

$$(x(t), y(t)) \equiv \text{ const}.$$

If $X(t)$ and $Y(t)$ are fundamental matrices of solutions of (6) and (12), respectively, then

$$Y^T(t)X(t) = C,$$

where C is a non-singular constant matrix.

5) The investigation of various special properties of linear systems, particularly the question of stability, is connected with the concept of the Lyapunov characteristic exponent of a solution and the first method in the theory of stability developed by A.M. Lyapunov (see Regular linear system; Reducible linear system; Lyapunov stability).

6) Two systems of the form (6) are said to be asymptotically equivalent if there is a one-to-one correspondence between their solutions $x_1(t)$ and $x_2(t)$ such that

$$\lim_{t \to \infty} (x_1(t) - x_2(t)) = 0.$$

If the system (11) with a constant matrix A is stable, then it is asymptotically equivalent to the system $\dot{x} = (A + B(t))x$, where the matrix $B(t)$ is continuous on $[a, +\infty)$ and

$$\int_0^\infty \|B(t)\| dt < \infty .$$

If (13) is satisfied, the system $\dot{x} = B(t)x$ is asymptotically equivalent to the system $\dot{x} = 0$.

Two systems of the form (11) with constant coefficients are said to be topologically equivalent if there is a homeomorphism $h: \mathbf{R}^n \to \mathbf{R}^n$ that takes oriented trajectories of one system into oriented trajectories of the other. If two square matrices A and B of order n have the same number of eigen values with negative real part and have no eigen values with zero real part, then the systems $\dot{x} = Ax$ and $\dot{x} = Bx$ are topologically equivalent.

7) Suppose that in the system (6) the matrix $A(t)$ is continuous and bounded on the whole real axis. The system (6) is said to have exponential dichotomy if the space $\mathbf{R}^n$ splits into a direct sum: $\mathbf{R}^n = \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}, n_1 + n_2 = n$, so that for every solution $x(t)$ with $x(0) \in \mathbf{R}^{n_1}$ the inequality

$$\|x(t)\| \geq ce^{k(t-t_0)}$$

holds, and for every solution $x(t)$ with $x(0) \in \mathbf{R}^{n_2}$ the inequality

$$\|x(t)\| \leq c^{-1}e^{-k(t-t_0)}$$

holds for all $t_0 \in \mathbf{R}$ and $t \geq t_0$, where $0 < c \leq 1$ and $k > 0$ are constants. For example, exponential dichotomy is present in a system (11) with constant matrix A if A has no eigen values with zero real part (such a system is said to be hyperbolic). If the vector function $b(t)$ is bounded on the whole real axis, then a system (5) having exponential dichotomy has a unique solution that is bounded on the whole line $\mathbf{R}$.

Check In Progress-II

Q. 1 Define linear ordinary differential equation.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q. 2 Define Wronski determinant.

Solution : . . . .
. . . .
. . . .

. . . . ……….. . . . .

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. . . .

2.4 DIFFERENTIAL EQUATION, ORDINARY

An equation with a function in one independent variable as unknown, containing not only the unknown function itself, but also its derivatives of various orders.

The term "differential equations" was was proposed in 1676 by G. Leibniz. The first studies of these equations were carried out in the late 17th century in the context of certain problems in mechanics and geometry.

Ordinary differential equations have important applications and are a powerful tool in the study of many problems in the natural sciences and in technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology. The reason for this is the fact that objective laws governing certain phenomena (processes) can be written as ordinary differential equations, so that the equations themselves are a quantitative expression of these laws. For instance, Newton's laws of mechanics make it possible to reduce the description of the motion of mass points or solid bodies to solving ordinary differential equations. The computation of radiotechnical circuits or satellite trajectories, studies of the stability of a plane in flight, and explaining the course of chemical reactions are all carried out by studying and solving ordinary differential equations. The most interesting and most important applications of these equations are in the theory of oscillations (cf. Oscillations, theory of) and in automatic control theory. Applied problems in turn produce new formulations of problems in the theory of ordinary differential equations; the mathematical theory of optimal control (cf. Optimal control, mathematical theory of) in fact arose in this manner.

Notes

In what follows the independent variable is denoted by t, the unknown functions by x, y, z, etc., while the derivatives of these functions with respect to t will be denoted by $\dot{x}, \ddot{x}, \dots, x^{(n)}$, etc.

The simplest ordinary differential equation is already encountered in analysis: The problem of finding the primitive function of a given continuous function $f(t)$ amounts to finding an unknown function $x(t)$ which satisfies the equation

$$\dot{x} = f(t) . \qquad (1)$$

In order to prove that this equation is solvable, a special apparatus had to be developed — the theory of the Riemann integral.

A natural generalization of equation (1) is an ordinary differential equation of the first order, solved with respect to the derivative:

$$\dot{x}(t) = f(t, x), \qquad (2)$$

where $f(t, x)$ is a known function, defined in a certain region of the (t, x)-plane. Many practical problems can be reduced to the solution (or, as is often said, the integration) of this equation. A solution of the ordinary differential equation (2) is a function $x(t)$ defined and differentiable on some interval I and satisfying the conditions

$$(t, x(t)) \in D, \qquad t \in I,$$

$$\dot{x}(t) = f(t, x(t)), \qquad t \in I .$$

The solution of (2) may be geometrically represented in the (t, x)-plane as a curve with equation $x = x(t)$, $t \in I$. This curve is known as an integral curve, with a tangent at every point, and is totally contained in D. The geometrical interpretation of equation (2) itself is as a field of directions in D, obtained by drawing a segment $l_{t,x}$ of small length with angular coefficient $f(t, x)$ through each point $(t, x) \in D$. Any integral curve $x = x(t)$ at each of its points is tangent to the segment $l_{t,x(t)}$.

The existence theorem answers the question of the existence of a solution of equation (2): If $f(t, x) \in C(D)$ (i.e. is continuous in D), then at least one continuously-differentiable integral curve of equation (2) passes through any point $(t_0, x_0) \in D$, and each such curve may be extended in both directions up to the boundary of any closed subregion lying completely in D and containing the point (t_0, x_0). In other words, for any point $(t_0, x_0) \in D$ it is possible to find at least one non-extendable solution $x = x(t)$, $t \in I$, such that $x(t) \in C^1(I)$ (i.e. x is continuous in I together with its derivative $\dot{x}$),

$$x(t_0) = x_0, \qquad (3)$$

and $x(t)$ tends to the boundary of D as t tends to the right or left end of the interval I.

A very important theoretical problem is to clarify the assumptions to be made concerning the right-hand side of an ordinary differential equation and the additional conditions to be imposed on the equation in order that it has a unique solution. The following existence and uniqueness theorem is valid: If $f(t, x) \in C(D)$ satisfies a Lipschitz condition with respect to x in D and if $(t_0, x_0) \in D$, then equation (2) has a unique, non-extendable solution satisfying condition (3). In particular, if two solutions $x_1(t)$, $t \in I_1$, and $x_2(t)$, $t \in I_2$, of such an equation (2) coincide for at least one value $t = t_0$, i.e. $x_1(t_0) = x_2(t_0)$, then

$$x_1(t) = x_2(t), \qquad t \in I_1 \bigcap I_2 .$$

The geometrical content of this theorem is that the entire region D is covered by integral curves of equation (2), with no intersections between any two curves. Unique solutions may also be obtained under weaker assumptions regarding the function $f(t, x)$.

The relation (3) is known as an initial condition. The numbers t_0 and x_0 are called initial values for the solution of equation (2), while the point (t_0, x_0) is called the initial point corresponding to the integral curve. The task of finding the solution of this equation satisfying initial condition (3) (or, in other words, with initial values t_0, x_0) is known as the Cauchy problem or the initial value problem. The theorem just given provides sufficient conditions for the unique solvability of the Cauchy problem (2), (3).

Notes

Applied problems often involve systems of ordinary differential equations, containing several unknown functions of the same variable and their derivatives. A natural generalization of equation (2) is the normal form of a system of differential equations of order n:

$$\dot{x}^{i} = f^{i}(t, x^{1}, ..., x^{n}), \quad i = 1, ..., n, \qquad (4)$$

where $x^{1}, ..., x^{n}$ are unknown functions of the variable t and f^{i}, $i = 1, ..., n$, are given functions in $n+1$ variables. Writing

$$\mathbf{x} = (x^{1}, ..., x^{n}),$$

$$\mathbf{f}(t, \mathbf{x}) = (f^{1}(t, \mathbf{x}), ..., f^{n}(t, \mathbf{x})),$$

the system (4) takes the vector form:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}). \qquad (5)$$

The vector function

$$\mathbf{x} = \mathbf{x}(t) = (x^{1}(t), ..., x^{n}(t)), \quad t \in I, \qquad (6)$$

is a solution of the system (4) or of the vector equation (5). Each solution can be represented in the $(n+1)$-dimensional space $t, x^{1}, ..., x^{n}$ as an integral curve — the graph of the vector function (6).

The Cauchy problem for equation (5) is to find the solution satisfying the initial conditions

$$x^{1}(t_0) = x_0^{1}, ..., x^{n}(t_0) = x_0^{n}, \qquad (7)$$

or

$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

The solution of the Cauchy problem (5), (7) is conveniently written as

$$\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0), \quad t \in I.$$

The existence and uniqueness theorem for equation (5) is formulated as for equation (2).

Very general systems of ordinary differential equations (solved with respect to the leading derivatives of all unknown functions) are reducible to normal systems. An important special class of systems (5) are linear systems of n (coupled) ordinary differential equations of the first order:

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{F}(t), \qquad (8)$$

where $A(t)$ is an $(n \times n)$-dimensional matrix.

Of major importance in applications and in the theory of ordinary differential equations are autonomous systems of ordinary differential equations (cf. Autonomous system):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad (9)$$

i.e. normal systems whose right-hand side does not explicitly depend on the variable t. In such a case equation (6) is conveniently regarded as a parametric representation of a curve, by regarding the solution as the phase trajectory in the n-dimensional phase space $x^1, \dots, x^n$.
If $\mathbf{x} = \mathbf{x}(t)$ is a solution of the system (9), the function $\mathbf{x} = \mathbf{x}(t + c)$, where c is an arbitrary constant, will also satisfy (9).

Another generalization of equation (2) is an ordinary differential equation of order n, solved with respect to its leading derivative:

$$y^{(n)} = f(t, y, \dot{y}, \dots, y^{(n-1)}) .$$

An important special class of such equations are linear ordinary differential equations:

$$y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_{n-1}(t)\dot{y} + a_n(t) y = F(t) . \qquad (10)$$

Equation (10) is reduced to a system of n first-order equations if one introduces new unknown functions of the variable t by the formulas

$$x^1 = y, x^2 = \dot{y}, \dots, x^n = y^{(n-1)} .$$

If, for example, equation (10) describes the dynamics of a certain object and the motion of this object is to be studied starting from a definite

Notes

moment $t = t_0$ corresponding to a definite initial state, the following additional conditions must be imposed on equation (10):

$$y(t_0) = y_0,\ \dot{y}(t_0) = \dot{y}_0\ ,...,\ y^{(n-1)}(t_0) = y_0^{(n-1)}. \qquad (11)$$

The task of finding an n times differentiable function $y = y(t)$, $t \in I$, for which equation (10) becomes an identity for all $t \in I$ and which satisfies the initial conditions (11) is known as the Cauchy problem.

The existence and uniqueness theorem: If

$$f(t, u_1\ ,..., u_n) \in C(D),$$

if it satisfies a Lipschitz condition with respect to $u_1\ ,..., u_n$ and if

$$(t_0, y_0, \dot{y}_0\ ,..., y_0^{(n-1)}) \in D,$$

then the Cauchy problem (10), (11) has a unique solution.

The Cauchy problem does not account for all problems which have been studied for equations (10) of higher orders (or systems (5)). Specific physical and technological problems often do not involve initial conditions but rather supplementary conditions of different kinds (so-called boundary conditions), when the values of the function $y(t)$ being sought and its derivatives (or relations between these derivatives) are given for certain different values of the independent variable. For instance, in the brachistochrone problem, the equation

$$2y\ddot{y} + \dot{y}\square^2 + 1 = 0$$

is to be integrated under the boundary conditions $y(a) = A$, $y(b) = B$. Finding a 2π-periodic solution for the Duffing equation is reduced to extracting the solution which satisfies the periodicity conditions $y(0) = y(2\pi)$, $\dot{y}(0) = \dot{y}(2\pi)$; in the study of laminar flow around a plate one encounters the problem:

$$\dddot{y} + y\ddot{y} = 0, \qquad y(0) = \dot{y}(0) = 0,$$

$$\dot{y}(t) \to 2 \quad \text{as } t \to \infty .$$

A problem of finding a solution satisfying conditions different from the initial conditions (11) for ordinary differential equations or for a system of ordinary differential equations is known as a boundary value problem (cf. Boundary value problem, ordinary differential equations). The theoretical analysis of the existence and uniqueness of a solution of a boundary value problem is of importance to the practical problem involved, since it proves the mutual compatibility of the assumptions made in the mathematical description of the problem and the relative completeness of this description. One important boundary value problem is the Sturm–Liouville problem. Boundary value problems for linear equations and systems are closely connected with problems involving eigen values and eigen functions (cf. Eigen function; Eigen value) and also with the spectral analysis of ordinary differential operators.

The principal task of the theory of ordinary differential equations is the study of solutions of such equations. However, the meaning of such a study of solutions of ordinary differential equations has been understood in various ways at different times. The original trend was to carry out the integration of equations in quadratures, i.e. to obtain a closed formula yielding (in explicit, implicit or parametric form) an expression for the dependence of a specific solution on t in terms of elementary functions and their integrals. Such formulas, if found, are of help in calculations and in the study of the properties of the solutions. Of special interest is the description of the totality of solutions of a given equation. Under very general assumptions, equation (5) corresponds to a family of vector functions depending on n arbitrary independent parameters. If the equation of this family has the form

$$\mathbf{x} = \phi(t, c_1, \dots, c_n),$$

the function ϕ is said to be the general solution of equation (5).

However, the first examples of ordinary differential equations which are not integrable in quadratures appeared in mid-19th century. It was found that solutions in closed form can be found for a few classes of equations only (see, for example, Bernoulli equation; Differential equation with total differential; Linear ordinary differential equation with constant coefficients). A detailed study was then begun of the most important and frequently encountered equations which cannot be solved in quadratures (e.g. the Bessel equation), special notation was introduced for such equations, their properties were studied and their values were tabulated. Many special functions appeared in this way.

Because of practical demands, methods of approximate integration of ordinary differential equations were also developed, such as the method of sequential approximation (cf. Sequential approximation, method of), the Adams method, etc. Various methods for graphical and mechanical integration of these equations were proposed. Mathematical analysis offers of a rich selection of numerical methods for solving many problems in ordinary differential equations (cf. Differential equations, ordinary, approximate methods of solution of). These methods are convenient computational algorithms with effective estimates of accuracy, and the modern computational techniques make it possible to obtain a numerical solution to each such problem in an economical and rapid manner.

However, the application of numerical methods to a specific equation yields only a finite number of particular solutions on a finite segment of variation of the independent variable. They cannot yield information about the asymptotic behaviour of the solutions, and cannot tell if a certain equation has a periodic solution or an oscillating solution. It is often important in many practical problems to establish the nature of the solution on an infinite interval of variation of the independent variable, and to obtain a complete picture of the integral curves. For this reason, the main trend in the theory of ordinary differential equations underwent a switchover to the study of the general features in the behaviour of solutions of ordinary differential equations, and to the development of methods for studying the global properties of solutions from the differential equation itself, without recourse to its integration.

All this formed the subject matter of the qualitative theory of differential equations, established in the late 19th century and still in full development.

Of decisive importance is the clarification as to whether or not the Cauchy problem is a well-posed problem for an ordinary differential equation. Since in concrete problems the initial values can never be perfectly exact, it is important to find the conditions under which small changes in initial values entail only small changes in the results. The theorem on continuous dependence of the solutions on initial values is valid: Let (8) be the solution of equation (5), where $\mathbf{f}(t, \mathbf{x}) \in C(D)$ and let it satisfy a Lipschitz condition with respect to $\mathbf{x}$; then, for any $\epsilon > 0$ and any compact $J \subset I$, $t_0 \in J$, it is possible to find a $\delta > 0$ such that the solution $\mathbf{x}(t, t_0, \mathbf{x}_0^*)$ of this equation, where $|\mathbf{x}_0^* - \mathbf{x}_0| < \delta$, is defined on J and for all $t \in J$,

$$|\mathbf{x}(t, t_0, \mathbf{x}_0^*) - \mathbf{x}(t, t_0, \mathbf{x}_0)| < \epsilon .$$

In other words, if the variations of the independent variable are restricted to a compact interval, then, if the variations in the initial values are sufficiently small, the solution will vary only slightly on the complete interval chosen. This result may also be generalized to obtain conditions which would ensure the differentiability of solutions (of differential equations) with respect to the initial values.

However, this theorem fails to give a complete answer to the problem which is of interest in practical applications, since it only speaks about a compact segment of variation of the independent variable. Now it is often necessary (e.g. in the theory of controlled motion) to deal with the solution of the Cauchy problem (5), (7) defined for all $t \geq t_0$, i.e. to clarify the stability of the solution with respect to small changes in the initial values on the entire infinite interval $t \geq t_0$, i.e. to obtain conditions which would ensure the validity of inequality (12) for all $t \geq t_0$. Studies of the stability of equilibrium positions or of the stationary conditions of a concrete system are reduced to this very problem. A solution which varies only to a small extent on the infinite interval $[t_0, \infty)$ if the deviations from the initial values are sufficiently small is said to be Lyapunov stable (cf. Lyapunov stability).

In selecting an ordinary differential equation to describe a real process, some features must always be be neglected and others idealized. This means that a description of a process by ordinary differential equations is only approximate. For instance, the study of the operation of a valve oscillator leads to the van der Pol equation if certain assumptions, which do not fully correspond to the real state of things, are made. Furthermore, the course of the process is often affected by perturbing factors which are practically impossible to allow for in setting up equations; all that is known is that their effect is "small" . It is therefore important to clarify the variation of the solution as a result of small variations in the system of equations itself, i.e. on passing from equation (5) to the perturbed equation

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}) + \mathbf{R}(t, \mathbf{x}),$$

which allows for small correction terms. It was found that on a compact interval the variations of the independent variable (under the same assumptions as in the theorem of continuous dependence of the solutions on the initial values) produce only small variations in the solution provided the perturbation $\mathbf{R}(t, \mathbf{x})$ is sufficiently small. If this property is retained on the infinite interval $t \geq t_0$, the solution is said to be stable under constantly acting perturbations.

Studies of Lyapunov stability, stability under constantly acting perturbations and their modifications form the subject of a highly important branch of the qualitative theory — stability theory. Of foremost interest in practice are systems of ordinary differential equations whose solutions change little for all small variations of these equations; such systems are known as robust systems (cf. Rough system).

Another important task in the qualitative theory is to obtain a pattern of the behaviour of the family of solutions throughout the domain of definition of the equation. In the case of the autonomous system (9) the problem is the construction of a phase picture, i.e. a qualitative overall description of the totality of phase trajectories in the phase space. Such a geometric picture gives a complete representation of the nature of all motions which may take place in the system under study. It is therefore important, first of all, to clarify the behaviour of the trajectories in a neighbourhood of equilibrium positions, and to find separatrices (cf. Separatrix) and limit cycles (cf. Limit cycle). An especially urgent task is to find stable limit cycles, since these correspond to auto-oscillations in real systems (cf. Auto-oscillation).

Any real object is characterized by different parameters, which often enter into the right-hand side of the system of ordinary differential equations describing the behaviour of the object,

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \epsilon),$$

in the form of certain quantities $(\epsilon^1, \dots, \epsilon^k) = \epsilon$. The values of these parameters are never known with perfect accuracy, so that it is important to clarify the conditions ensuring the stability of the solutions of equation (13) to small perturbations of the parameter ϵ. If the independent variable varies in a given compact interval, then — under certain natural assumptions regarding the right-hand side of equation (13) — the solutions will show a continuous (and even differentiable) dependence on the parameters.

The clarification of the dependence of the solutions on the parameter is directly related to the question of the quality of the idealization leading to the mathematical model of the behaviour of the object — the system of ordinary differential equations. A typical example of idealization is the neglect of a small parameter. If, with allowance for this small parameter, the system (13) is obtained, then, owing to the fact that the variation of the solutions with the parameter is continuous, it is perfectly permissible to neglect this parameter in the study of the behaviour of the object on a compact interval of time. Thus, as a first approximation, one is considering the simpler system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, 0) .$$

This result is the principle of the extensively employed method of small parameters (cf. Small parameter, method of the); the Krylov–Bogolyubov method of averaging and other asymptotic methods for solving ordinary differential equations. However, the study of a number of phenomena yields a system of differential equations with small parameter in front of the derivative:

$$\epsilon\dot{x} = f(t, x, y), \qquad \dot{y} = g(t, x, y) .$$

Notes

Here it is in general no longer permissible to assume that $\epsilon = 0$, even if it is attempted to construct a rough representation of the phenomenon on a compact interval of time.

The theory of ordinary differential equations considers certain fruitful important generalizations of the problems outlined above. First, one may extend the class of functions within which the solution of the Cauchy problem (2), (3) is sought: Determine the solution in the class of absolutely-continuous functions and prove the existence of such solutions. Of special practical interest is to find the solution of equation (2) if the function $f(t, x)$ is discontinuous or many-valued with respect to x. The most general problem in this respect is the problem of solving a differential inclusion.

Also under consideration are ordinary differential equations of order n more general than (10), which are unsolved with respect to the leading derivative

$$F(t, y, \dot{y}, \dots, y^{(n)}) = 0 .$$

Studies of this equation are closely connected with the theory of implicit functions.

Equation (2) connects the derivative of the solution at a point t with the value of the solution at this point: $\dot{x}(t) = f(t, x(t))$, but certain applied problems (e.g. those in which allowance must be made for a delaying effect of the executing mechanism) yield retarded ordinary differential equations (cf. Differential equations, ordinary, retarded):

$$\dot{x} = f(t, x(t - \tau)),$$

in which the derivative of the solution at a point t is connected with the value of the solution at a point $t - \tau$. A special section of the theory of ordinary differential equations deals with such equations, and also with the more general ordinary differential equations with distributed arguments (cf. Differential equations, ordinary, with distributed arguments).

The study of the phase space of the autonomous system (9) leads to yet another generalization of ordinary differential equations. Denote by $\mathbf{x}=\mathbf{x}(t,\mathbf{x}_0)$ the trajectory of this system passing through the point $\mathbf{x}_0$. If the point $\mathbf{x}_0$ is mapped to the point $\mathbf{x}(t,\mathbf{x}_0)$, one obtains a transformation of the phase space depending on the parameter t which determines the motion in this space. The properties of such motions are studied in the theory of dynamical systems. They may be studied not only in Euclidean space but also on manifolds; an example are differential equations on a torus.

Above ordinary differential equations in the field of real numbers have been considered (e.g. finding a real-valued function $x(t)$ of a real variable t satisfying equation (2)). However, certain properties of such equations are more conveniently studied with the aid of complex numbers. A natural further generalization is the study of ordinary differential equations in the field of complex numbers. Thus, one may consider the equation

$$\frac{dw}{dz}=f(z,w),$$

where $f(z,w)$ is an analytic function of its variables, and pose the problem of finding an analytic function $w(z)$ in the complex variable z which would satisfy this equation. The study of such equations, equations of higher orders and systems forms the subject of the analytic theory of differential equations; in particular, it contains results of importance to mathematical physics, concerning linear ordinary differential equations of the second order (cf. Linear ordinary differential equation of the second order).

One may also consider the equation

$$\frac{d\mathbf{x}}{dt}=\mathbf{f}(t,\mathbf{x})$$

on the assumption that $\mathbf{x}$ belongs to an infinite-dimensional Banach space B, t is a real or complex independent variable and $\mathbf{f}(t,\mathbf{x})$ is an operator mapping the product $(-\infty,+\infty)\times B$ into B. Equation (14) may serve in processing, for example, systems of ordinary differential equations of infinite order (cf. Differential equations, infinite-order system of). Equations of the type (14) are studied in the theory of abstract differential equations (cf. Differential equation, abstract), which is the meeting point of ordinary differential equations and functional analysis. Of major interest are linear differential equations of the form

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{F}(t)$$

with bounded or unbounded operators; certain classes of partial differential equations (cf. Differential equation, partial) can be written in the form of such an equation.

2.5 SUMMARY

In this unit we study

Given a first-order ordinary differential equation

$$\frac{dy}{dx} = F(x, y),$$

if $F(x, y)$ can be expressed using separation of variables as

$$F(x, y) = X(x)\,Y(y),$$

The Cauchy problem is to find the solution satisfying the initial conditions

$$x^1(t_0) = x_0^1, \ldots, x^n(t_0) = x_0^n,$$

A first approximation, one is considering the simpler system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, 0) \; .$$

This result is the principle of the extensively employed method of small parameters.

2.6 KEYWORD

Infinite-Dimensional : This approach consists in the extension of the definition of small and large inductive dimensions to infinite ordinal numbers

Dynamical Systems : A dynamical system is a system in which a function describes the time dependence of a point in a geometrical space

Bernoulli equation : The Bernoulli Equation can be considered to be a statement of the conservation of energy principle appropriate for flowing fluids. The qualitative behavior that is usually labeled with the term "Bernoulli effect" is the lowering of fluid pressure in regions where the flow velocity is increased.

2.7 EXERCISE

Q. 1 Define linear ordinary differential equation.

Q. 2 Define first-order ordinary differential equation.

Q. 3 Define Cauchy Problem of ordinary differential equation.

Q. 4 Solve fundamental matrix system.

2.8 ANSWER TO CHECK IN PROGRESS

Check In Progress-1

Answer Q. 1 Check in section 3.1

Q. 2 Check in section 3

Check In progress-II

Answer Q. 1 Check in section 4

Q. 2 Check in section 4

2.9 SUGGESTION READING AND REFERENCES

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UNIT 3 SECOND ODER DIFFERENTIAL EQUATIONS

STRUCTURE

3.0 OBJECTIVE

- In this unit we study second order differential equation with its properties and examples.
- We study basic concept of second order ordinary differential equation. We study In general, little is known about nonlinear second order differential equations

$$y'' = f(x, y, y').,$$

- We study Linear Second Order Differential Equations
- We shall also study Method of Undetermined Coefficient or Guessing Method

3.1 INTRODUCTION

A second order differential equation is an equation involving the unknown function *y*, its derivatives *y*' and *y*'', and the variable *x*. We will only consider explicit differential equations of the form,

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

3.2 SECOND ORDER DIFFERENTIAL EQUATIONS

In the previous chapter we looked at first order differential equations. In this chapter we will move on to second order differential equations. Just as we did in the last chapter we will look at some special cases of second order differential equations that we can solve. Unlike the previous chapter however, we are going to have to be even more restrictive as to the kinds of differential equations that we'll look at. This will be required in order for us to actually be able to solve them.

Here is a list of topics that will be covered in this chapter.

Basic Concepts – In this section give an in depth discussion on the process used to solve homogeneous, linear, second order differential equations, ay″+by′+cy=0ay″+by′+cy=0. We derive the characteristic polynomial and discuss how the Principle of Superposition is used to get the general solution.

Real Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, ay″+by′+cy=0ay″+by′+cy=0, in which the roots of the characteristic polynomial, ar2+br+c=0ar2+br+c=0, are real distinct roots.

Complex Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, ay″+by′+cy=0ay″+by′+cy=0, in which the roots of the characteristic polynomial, ar2+br+c=0ar2+br+c=0, are complex roots. We will also derive from the complex roots the standard solution that is typically used in this case that will not involve complex numbers.

Repeated Roots – In this section we discuss the solution to homogeneous, linear, second order differential equations, ay″+by′+cy=0ay″+by′+cy=0, in which the roots of the characteristic polynomial, ar2+br+c=0ar2+br+c=0, are repeated, *i.e.* double, roots. We will use reduction of order to derive the second solution needed to get a general solution in this case.

Reduction of Order – In this section we will take a brief look at the topic of reduction of order. This will be one of the few times in this chapter that non-constant coefficient differential equation will be looked at.

Fundamental Sets of Solutions – In this section we will a look at some of the theory behind the solution to second order differential equations. We define fundamental sets of solutions and discuss how they can be used to get a general solution to a homogeneous second order differential equation. We will also define the Wronskian and show how it can be used to determine if a pair of solutions are a fundamental set of solutions.

More on the Wronskian – In this section we will examine how the Wronskian, introduced in the previous section, can be used to determine if two functions are linearly independent or linearly dependent. We will also give and an alternate method for finding the Wronskian.

Nonhomogeneous Differential Equations – In this section we will discuss the basics of solving nonhomogeneous differential equations. We define the complimentary and particular solution and give the form of the general solution to a nonhomogeneous differential equation.

Undetermined Coefficients – In this section we introduce the method of undetermined coefficients to find particular solutions to nonhomogeneous differential equation. We work a wide variety of examples illustrating the many guidelines for making the initial guess of the form of the particular solution that is needed for the method.

Variation of Parameters – In this section we introduce the method of variation of parameters to find particular solutions to nonhomogeneous differential equation. We give a detailed examination of the method as well as derive a formula that can be used to find particular solutions.

Mechanical Vibrations – In this section we will examine mechanical vibrations. In particular we will model an object connected to a spring and moving up and down. We also allow for the introduction of a damper to the system and for general external forces to act on the object. Note as well that while we example mechanical vibrations in this section a simple change of notation (and corresponding change in what the quantities represent) can move this into almost any other engineering field.

3.3 SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION

An ordinary differential equation of the form

$$y'' + P(x)y' + Q(x)y = 0. \tag{1}$$

Such an equation has singularities for finite $x = x_0$ under the following conditions: (a) If either $P(x)$ or $Q(x)$ diverges as $x \to x_0$,
but $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ remain finite as $x \to x_0$, then x_0 is called

a regular or nonessential singular point. (b) If $P(x)$ diverges faster than $(x-x_0)^{-1}$ so that $(x-x_0)P(x)\to\infty$ as $x\to x_0$, or $Q(x)$ diverges faster than $(x-x_0)^{-2}$ so that $(x-x_0)^2 Q(x)\to\infty$ as $x\to x_0$, then x_0 is called an irregular or essential singularity.

Singularities of equation (1) at infinity are investigated by making the substitution $x\equiv z^{-1}$, so $dx=-z^{-2}\,dz$, giving

$$\frac{dy}{dx}=-z^2\frac{dy}{dz} \qquad (2)$$

$$\frac{d^2y}{dx^2}=-z^2\frac{d}{dz}\left(-z^2\frac{dy}{dz}\right) \qquad (3)$$

$$=-z^2\left(-2z\frac{dy}{dz}-z^2\frac{d^2y}{dz^2}\right) \qquad (4)$$

$$=2z^3\frac{dy}{dz}+z^4\frac{d^2y}{dz^2}. \qquad (5)$$

Then (3) becomes

$$z^4\frac{d^2y}{dz^2}+\left[2z^3-z^2P\left(z^{-1}\right)\right]\frac{dy}{dz}+Q\left(z^{-1}\right)y=0. \qquad (6)$$

Case (a): If

$$\alpha(z)\equiv\frac{2z-P\left(z^{-1}\right)}{z^2} \qquad (7)$$

$$\beta(z)\equiv\frac{Q\left(z^{-1}\right)}{z^4} \qquad (8)$$

remain finite at $x=\pm\infty$ ($z=0$), then the point is ordinary. Case (b): If either $\alpha(z)$ diverges no more rapidly than $1/z$ or $\beta(z)$ diverges no more rapidly than $1/z^2$, then the point is a regular singular point. Case (c): Otherwise, the point is an irregular singular point.

Morse and Feshbach (1953, pp. 667-674) give the canonical forms and solutions for second-order ordinary differential equations classified by types of singular points.

For special classes of linear second-order ordinary differential equations, variable coefficients can be transformed into constant coefficients. Given a second-order linear ODE with variable coefficients

$$\frac{d^2 y}{d x^2} + p(x) \frac{d y}{d x} + q(x) y = 0. \tag{9}$$

Define a function $z \equiv y(x)$,

$$\frac{d y}{d x} = \frac{d z}{d x} \frac{d y}{d z} \tag{10}$$

$$\frac{d^2 y}{d x^2} = \left(\frac{d z}{d x}\right)^2 \frac{d^2 y}{d z^2} + \frac{d^2 z}{d x^2} \frac{d y}{d z} \tag{11}$$

$$\left(\frac{d z}{d x}\right)^2 \frac{d^2 y}{d z^2} + \left[\frac{d^2 z}{d x^2} + p(x) \frac{d z}{d x}\right] \frac{d y}{d z} + q(x) y = 0 \tag{12}$$

$$\frac{d^2 y}{d z^2} + \left[\frac{\frac{d^2 z}{d x^2} + p(x) \frac{d z}{d x}}{\left(\frac{d z}{d x}\right)^2}\right] \frac{d y}{d z} + \left[\frac{q(x)}{\left(\frac{d z}{d x}\right)^2}\right] y \tag{13}$$

$$\equiv \frac{d^2 y}{d z^2} + A \frac{d y}{d z} + B y = 0. \tag{14}$$

This will have constant coefficients if A and B are not functions of x. But we are free to set B to an arbitrary positive constant for $q(x) \geq 0$ by defining z as

$$z \equiv B^{-1/2} \int [q(x)]^{1/2} \, d x. \tag{15}$$

Then

$$\frac{d z}{d x} = B^{-1/2} [q(x)]^{1/2} \tag{16}$$

$$\frac{d^2 z}{d x^2} = \frac{1}{2} B^{-1/2} [q(x)]^{-1/2} q'(x), \tag{17}$$

and

$$A = \frac{\frac{1}{2} B^{-1/2} [q(x)]^{-1/2} q'(x) + B^{-1/2} p(x) [q(x)]^{1/2}}{B^{-1} q(x)} \tag{18}$$

$$= \frac{q'(x) + 2 p(x) q(x)}{2 [q(x)]^{3/2}} B^{1/2}. \tag{19}$$

Equation (◇) therefore becomes

$$\frac{d^2 y}{d z^2} + \frac{q'(x) + 2 p(x) q(x)}{2 [q(x)]^{3/2}} B^{1/2} \frac{d y}{d z} + B y = 0, \tag{20}$$

which has constant coefficients provided that

$$A \equiv \frac{q'(x) + 2\,p(x)\,q(x)}{2\,[q(x)]^{3/2}}\,B^{1/2} = [\text{constant}]. \tag{21}$$

Eliminating constants, this gives

$$A' \equiv \frac{q'(x) + 2\,p(x)\,q(x)}{[q(x)]^{3/2}} = [\text{constant}]. \tag{22}$$

So for an ordinary differential equation in which A' is a constant, the solution is given by solving the second-order linear ODE with constant coefficients

$$\frac{d^2 y}{d z^2} + A\,\frac{d y}{d z} + B\,y = 0 \tag{23}$$

for z, where z is defined as above.

A linear second-order homogeneous differential equation of the general form

$$y'' + P(x)\,y' + Q(x)\,y = 0 \tag{24}$$

can be transformed into standard form

$$z'' + q(x)\,z = 0 \tag{25}$$

with the first-order term eliminated using the substitution

$$\ln y \equiv \ln z - \tfrac{1}{2}\int P(x)\,d x. \tag{26}$$

Then

$$\frac{y'}{y} = \frac{z'}{z} - \tfrac{1}{2}\,P(x) \tag{27}$$

$$\frac{y\,y'' - y'^2}{y^2} = \frac{z\,z'' - z'^2}{z^2} - \tfrac{1}{2}\,P'(x) \tag{28}$$

$$\frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2}\,P'(x) \tag{29}$$

$$\frac{y''}{y} = \left[\frac{z'}{z} - \tfrac{1}{2}\,P(x)\right]^2 + \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2}\,P'(x) \tag{30}$$

$$= \frac{z'^2}{z^2} - \frac{z'}{z}\,P(x) + \tfrac{1}{4}\,P^2(x) + \frac{z''}{z} - \frac{z'^2}{z^2} - \tfrac{1}{2}\,P'(x), \tag{31}$$

so

$$\frac{y''}{y} + P(x) = -\frac{z'}{z}P(x) + \frac{1}{4}P^2(x) + \frac{z''}{z} - \frac{1}{2}P'(x) + P(x)\left[\frac{z'}{z} - \frac{1}{2}P(\right. \tag{32}$$

$$= \frac{z''}{z} - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x) + Q(x) = 0. \tag{33}$$

Therefore,

$$z'' + \left[Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x)\right]z \equiv z''(x) + q(x)z = 0, \tag{34}$$

where

$$q(x) \equiv Q(x) - \frac{1}{2}P'(x) - \frac{1}{4}P^2(x). \tag{35}$$

If $Q(x) = 0$, then the differential equation becomes

$$y'' + P(x)y' = 0, \tag{36}$$

which can be solved by multiplying by

$$\exp\left[\int^x P(x')\,dx'\right] \tag{37}$$

to obtain

$$0 = \frac{d}{dx}\left\{\exp\left[\int^x P(x')\,dx'\right]\frac{dy}{dx}\right\} \tag{38}$$

$$c_1 = \exp\left[\int^x P(x')\,dx'\right]\frac{dy}{dx} \tag{39}$$

$$y = c_1\int^x \frac{dx}{\exp\left[\int^x P(x')\,dx'\right]} + c_2. \tag{40}$$

For a nonhomogeneous second-order ordinary differential equation in which the x term does not appear in the function $f(x, y, y')$,

$$\frac{d^2 y}{dx^2} = f(y, y'), \tag{41}$$

let $v \equiv y'$, then

$$\frac{dv}{dx} = f(v, y) = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}. \tag{42}$$

So the first-order ODE

$$v \frac{dv}{dy} = f(y, v), \tag{43}$$

if linear, can be solved for v as a linear first-order ODE. Once the solution is known,

$$\frac{dy}{dx} = v(y) \tag{44}$$

$$\int \frac{dy}{v(y)} = \int dx. \tag{45}$$

On the other hand, if y is missing from $f(x, y, y')$,

$$\frac{d^2 y}{dx^2} = f(x, y'), \tag{46}$$

let $v \equiv y'$, then $v' = y''$, and the equation reduces to

$$v' = f(x, v), \tag{47}$$

which, if linear, can be solved for v as a linear first-order ODE. Once the solution is known,

$$y = \int v(x)\, dx. \tag{48}$$

Nonhomogeneous ordinary differential equations can be solved if the general solution to the homogenous version is known, in which case variation of parameters can be used to find the particular solution. In particular, the particular solution $y^*(x)$ to a nonhomogeneous second-order ordinary differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{49}$$

can be found using variation of parameters to be given by the equation

$$y^*(x) = -y_1(x) \int \frac{y_2(x)\, g(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x)\, g(x)}{W(x)}\, dx, \tag{50}$$

where $y_1(x)$ and $y_2(x)$ are the homogeneous solutions to the unforced equation

$$y'' + p(x)y' + q(x)y = 0 \tag{51}$$

and $W(x)$ is the Wronskian of these two functions.

Check In Progress-I

Q. 1 Define second-order ordinary differential equation.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q. 2 Define non-homogeneous Differential Equation.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

3.4 NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

In general, little is known about nonlinear second order differential equations

$$y'' = f(x, y, y').,$$

but two cases are worthy of discussion:

(1) Equations with the *y* missing

$$y'' = f(x, y').$$

Let $v = y'$. Then the new equation satisfied by v is

$$v' = f(x, v).$$

This is a first order differential equation. Once *v* is found its integration gives the function *y*.

Example 1: Find the solution of

$$xy'' + 2y' + x = 1\ , \quad y(1) = 2\ , \quad y'(1) = 1\ .$$

Solution: Since *y* is missing, set *v*=*y*'. Then, we have

$$xv' + 2v + x = 1\ .$$

This is a first order linear differential equation. Its resolution gives

$$v = \frac{c}{x^2} + \frac{1}{2} - \frac{x}{3}\ .$$

Since $v(1) = 1$, we get $c = \frac{5}{6}$. Consequently, we have

$$v = \frac{5}{6x^2} + \frac{1}{2} - \frac{x}{3}\ .$$

Since *y*'=*v*, we obtain the following equation after integration

$$y = C - \frac{5}{6x} + \frac{x}{2} - \frac{x^2}{6}\ .$$

The condition $y(1) = 2$ gives $C = \frac{5}{2}$. Therefore, we have

$$y = \frac{5}{2} - \frac{5}{6x} + \frac{x}{2} - \frac{x^2}{6}\ .$$

Note that this solution is defined for $x > 0$.

(2)Equations with the *x* missing

$$y'' = f(y, y').$$

Let $v = y'$. Since

Notes

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy},$$

we get

$$v\frac{dv}{dy} = f(y, v).$$

This is again a first order differential equation. Once *v* is found then we can get *y* through

$$y' = v(y)$$

which is a separable equation. Beware of the constants solutions.

Example 2: Find the general solution of the equation

$$y'' + (y')^3 y = 0 \ .$$

Solution: Since the variable *x* is missing, set *v*=*y*'. The formulas above lead to

$$v\frac{dv}{dy} + v^3 y = 0 \ .$$

This a first order separable differential equation. Its resolution gives

$$\begin{cases} v = 0 \\ v = \dfrac{2}{y^2 - 2C} \end{cases} .$$

Since $y' = \frac{dy}{dx} = v$, we get *y*' = 0 or

$$\frac{dy}{dx} = \frac{2}{y^2 - 2C} \ .$$

Since this is a separable first order differential equation, we get, after resolution,

$$\frac{y^3}{3} - 2Cy = 2x + C^*,$$

where C and C^* are two constants. All the solutions of our initial equation are

$$\begin{cases} y = C \\ \dfrac{y^3}{3} - 2Cy = 2x + C^* . \end{cases}$$

Note that we should pay special attention to the constant solutions when solving any separable equation. This may be source of mistakes...

3.5 LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

A linear second order differential equations is written as

$$a(x)y'' + b(x)y' + c(x)y = d(x).$$

When d(x) = 0, the equation is called homogeneous, otherwise it is called nonhomogeneous. To a nonhomogeneous equation

$$(NH) \qquad a(x)y'' + b(x)y' + c(x)y = d(x).,$$

we associate the so called associated homogeneous equation

$$(H) \qquad a(x)y'' + b(x)y' + c(x)y = 0.$$

For the study of these equations we consider the explicit ones given by

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x) \\ y'' + p(x)y' + q(x)y = 0 \end{cases}$$

where p(x) = b(x)/a(x), q(x) = c(x)/a(x) and g(x) = d(x)/a(x). If p(x), q(x) and g(x) are defined and continuous on the interval I, then the IVP

$$y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = \alpha, \quad y'(x_0) = \beta,$$

where $x_0 \in I$ and α, β are arbitrary numbers, has a unique solution defined on I.

Main result: The general solution to the equation (NH) is given by

$$y = y_h + y_p,$$

where

(i) y_h is the general solution to the homogeneous associated equation (H);

(ii) y_p is a particular solution to the equation (NH).

In conclusion, we deduce that in order to solve the nonhomogeneous equation (NH), we need to

Step 1: find the general solution to the homogeneous associated equation (H), say y_h;

Step 2: find a particular solution to the equation (NH), say y_p;

Step 3: write down the general solution to (NH) as

$$y = y_h + y_p .$$

3.5.1 Homogeneous Linear Equations

Consider the homogeneous second order linear equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

or the explicit one

$$y'' + p(x)y' + q(x)y = 0 .$$

Basic property:*If* y_1 *and* y_2 *are two solutions, then*

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution for any arbitrary constants c_1, c_2.

The natural question to ask is whether any solution *y* is equal to $c_1 y_1 + c_2 y_2$ for some c_1 and c_2.

3.5.2 Linear Independence and the Wronskian

Let y_1 and y_2 be two differentiable functions.

The **Wronskian** $W(y_1, y_2)$, associated to y_1 and y_2, is the function

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

We have the following important properties:

(1) If y_1 and y_2 are two solutions of the equation y" + p(x)y' + q(x)y = 0, then

$$W(y_1, y_2)(x) = W(y_1, y_2)(x_0)\exp\left(-\int_{x_0}^{x} p(t)dt\right).$$

(2) If y_1 and y_2 are two solutions of the equation y" + p(x)y' + q(x)y = 0, then

$$W(y_1, y_2)(x) \neq 0 \text{ for every } x \iff \exists x_0 \text{ such that } W(y_1, y_2)(x_0) \neq 0 .$$

In this case, we say that y_1 and y_2 are linearly independent.

(3) If y_1 and y_2 are two linearly independent solutions of the equation y" + p(x)y' + q(x)y = 0, then any solution y is given by

$$y = c_1 y_1 + c_2 y_2$$

for some constant c_1 and c_2. In this case, the set $\{y_1, y_2\}$ is called the **fundamental set** of solutions.

Example: Let y_1 be the solution to the IVP

$$y'' + (2x - 1)y' + \sin(e^x)y = 0 \quad y(0) = 1, \; y'(0) = -1 ;$$

and y_2 be the solution to the IVP

$$y'' + (2x - 1)y' + \sin(e^x)y = 0 \quad y(0) = 2, \; y'(0) = 1 .$$

Find the Wronskian of $\{y_1, y_2\}$. Deduce the general solution to

Notes

$$y'' + (2x - 1)y' + \sin(e^x)y = 0 \ .$$

Solution: Let us write $W(x) = W(y_1, y_2)(x)$. We know from the properties that

$$W(x) = W(0)e^{-\int_0^x (2t-1)dt} = W(0)e^{-x^2 + x} \ .$$

Let us evaluate W(0). We have

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 1 + 2 = 3 \ .$$

Therefore, we have

$$W(x) = 3e^{-x^2+x} \ .$$

Since $W(0) \neq 0$, we deduce that $\{y_1, y_2\}$ is a fundamental set of solutions. Therefore, the general solution is given by

$$y = c_1 y_1 + c_2 y_2,$$

where c_1, c_2 are arbitrary constants.

3.5.3 Reduction of Order Technique

This technique is very important since it helps one to find a second solution independent from a known one. Therefore, according to the previous section, in order to find the general solution to $y'' + p(x)y' + q(x)y = 0$, we need only to find one (non-zero) solution, y_1.

Let y_1 be a non-zero solution of

$$y'' + p(x)y' + q(x)y = 0.$$

Then, a second solution y_2 independent of y_1 can be found as

$$y_2(x) = y_1(x)v(x) \ .$$

Easy calculations give

$$v(x) = C \int \frac{1}{y_1^2(x)} e^{-\int p(x)dx} \, dx ,$$

where *C* is an arbitrary non-zero constant. Since we are looking for a second solution one may take *C*=1, to get

$$y_2 = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int p(x)dx} \, dx \; .$$

Remember that this formula saves time. But, if you forget it you will have to plug $y_1 v(x)$ into the equation to determine *v*(*x*) which may lead to mistakes !

The general solution is then given by

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Example: Find the general solution to the Legendre equation

$$(1 - x^2)y'' - 2xy' + 2y = 0 \quad -1 < x < 1,$$

using the fact that $y_1 = x$ is a solution.

Solution: It is easy to check that indeed $y_1 = x$ is a solution. First, we need to rewrite the equation in the explicit form

$$y'' - \frac{2x}{1 - x^2} y' + \frac{2}{1 - x^2} y = 0 \; .$$

We may try to find a second solution $y_2 = xv(x)$ by plugging it into the equation. We leave it to the reader to do that! Instead let us use the formula

$$y_2 = x \int \frac{1}{x^2} e^{-\int -2x/(1-x^2)dx} \, dx \; .$$

Techniques of integration (of rational functions) give

Notes

$$\int \frac{1}{x^2} e^{\left(\int \frac{2x}{(1-x^2)} dx\right)} dx = \int \frac{1}{x^2(1-x^2)} dx = -\frac{1}{x} + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
,

which gives

$$y_2(x) = -1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) .$$

The general solution is then given by

$$y = c_1 x + c_2\left(-1 + \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right) .$$

Check In Progress-II

Q. 1 Find the general solution of the equation

$$y'' + (y')^3 y = 0 .$$

Solution : . . . .
. . . .
. . . .
. . . .

Q. 2 Find the general solution to the Legendre equation

$$(1-x^2)y'' - 2xy' + 2y = 0 \quad -1 < x < 1,$$

using the fact that $y_1 = x$ is a solution.

Solution : . . . .
. . . .
. . . .
. . . .

3.6 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

A second order homogeneous equation with constant coefficients is written as

$$ay'' + by' + cy = 0 \qquad (a \neq 0)$$

where a, b and c are constant. This type of equation is very useful in many applied problems (physics, electrical engineering, etc..). Let us summarize the steps to follow in order to find the general solution:

(1) Write down the characteristic equation

$$ar^2 + br + c = 0.$$

This is a quadratic equation. Let r_1 and r_2 be its roots (we have $r_{1,2} = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a$);

(2) If r_1 and r_2 are distinct real numbers (this happens if $b^2 - 4ac > 0$), then the general solution is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

(3) If $r_1 = r_2$ (which happens if $b^2 - 4ac = 0$), then the general solution is

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}.$$

(4) If r_1 and r_2 are complex numbers (which happens if $b^2 - 4ac < 0$), then the general solution is

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x),$$

where

$$r_1 = \overline{r_2} = \alpha + i\beta = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a},$$

that is,

Notes

$$\alpha = \frac{-b}{2a} \text{ and } \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

Example: Find the solution to the IVP

$$y'' + 2y' + 2y = 0 \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2.$$

Solution: Let us follow the steps:

1 Characteristic equation and its roots

$$r^2 + 2r + 2 = 0.$$

Since 4-8 = -4<0, we have complex roots $-1 \pm i$.

Therefore, $\alpha = -1$ and $\beta = 1$;

2 General solution

$$y = c_1 e^{-x} \cos(x) + c_2 e^{-x} \sin(x);$$

3 In order to find the particular solution we use the initial conditions to determine c_1 and c_2. First, we have

$$y(\pi/4) = c_1 e^{-\pi/4} \frac{\sqrt{2}}{2} + c_2 e^{-\pi/4} \frac{\sqrt{2}}{2}.$$

Since

$y'(x) = c_1(-\cos(x) - \sin(x))e^{-x} + c_2(-\sin(x) + \cos(x))e^{-x}$

, we get

$$y'(\pi/4) = c_1\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)e^{-\pi/4}.$$

From these two equations we get

$$c_1 = c_2 = \sqrt{2}e^{\pi/4},$$

which implies

$$y = \left(\cos(x)e^{-x} + \sin(x)e^{-x}\right)\sqrt{2}e^{\pi/4}\ .$$

3.6.1 Non-homogeneous Second Order Linear Equations

Let us go back to the non-homogeneous second order linear equations

$$(NH) \qquad y'' + p(x)y' + q(x)y = g(x)\ .$$

Recall that the general solution is given by

$$y = y_p(x) + y_h(x)$$

where $y_p(x)$ is a particular solution of (NH) and $y_h(x)$ is the general solution of the associated homogeneous equation

$$(H) \qquad y'' + p(x)y' + q(x)y = 0.$$

In the previous sections we discussed how to find y_h. In this section we will discuss two major techniques giving y_p:

3.6.2 Method of Undetermined Coefficient or Guessing Method

This method is based on a guessing technique. That is, we will guess the form of y_p and then plug it in the equation to find it. However, it works only under the following two conditions:

Condition 1: the associated homogeneous equations has constant coefficients;

Condition 2: the nonhomogeneous term $g(x)$ is a special form

$$g(x) = P(x)e^{\alpha x}\cos(\beta x), \quad \text{or} \quad g(x) = L(x)e^{\alpha x}\sin(\beta x)\ .$$

where $P(x)$ and $L(x)$ are polynomial functions.

Note that we may assume that $g(x)$ is a sum of such functions (see the remark below for more on this).

Assume that the two conditions are satisfied. Consider the equation

Notes

$$ay'' + by' + cy = g(x),$$

where a, b and c are constants and

$$g(x) = P_n(x)e^{\alpha x}\cos(\beta x), \quad \text{or} \quad g(x) = P_n(x)e^{\alpha x}\sin(\beta x),$$

where $P_n(x)$ is a polynomial function with degree n. Then a particular solution y_p is given by

$$y_p(x) = x^s\left(T_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)\right),$$

where

$$T_n(x) = A_0 + A_1 x + \dots + A_n x^n, \quad \text{and} \quad R_n(x) = B_0 + B_1 x + \dots + B_n x^n$$
,

where the constants A_k and B_k have to be determined. The power s is equal to 0 if $\alpha + i\beta$ is not a root of the characteristic equation. If $\alpha + i\beta$ is a simple root, then s=1 and s=2 if it is a double root.

Remark: If the nonhomogeneous term $g(x)$ satisfies the following

$$g(x) = \sum_{i=1}^{i=N} g_i(x),$$

where $g_i(x)$ are of the forms cited above, then we split the original equation into N equations

$$ay'' + by' + cy = g_i(x) \quad (i = 1, 2, .., N)$$

then find a particular solution y_i. A particular solution to the original equation is given by

$$y_p(x) = \sum_{i=1}^{i=N} y_i.$$

Summary:Let us summarize the steps to follow in applying this method:

(1)First, check that the two conditions are satisfied;

(2)If the equation is given as

$$ay'' + by' + cy = \sum_{k=1}^{k=N} g_k(x),$$

where $g_k(x) = P_n(x)e^{\alpha_k x}\cos(\beta_k x)$ or $g_k(x) = P_n(x)e^{\alpha_k x}\sin(\beta_k x)$ where P_n is a polynomial function with degree n, then split this equation into N equations

$$ay'' + by' + cy = g_k(x);$$

(3)Write down the characteristic equation $ar^2 + br + c = 0$, and find its roots;

(4)Write down the number $\alpha_k + i\beta_k$. Compare this number to the roots of the characteristic equation found in previous step.

(4.1)If $\alpha_k + i\beta_k$ is not one of the roots, then set s = 0;

(4.2)If $\alpha_k + i\beta_k$ is one of the two distinct roots, set s = 1;

(4.3)If $\alpha_k + i\beta_k$ is equal to both root (which means that the characteristic equation has a double root), set s=2.

In other words, s measures how many times $\alpha_k + i\beta_k$ is a root of the characteristic equation;

(5)Write down the form of the particular solution

$$y_k = x^s\left(T_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)\right),$$

where

$$T_n(x) = A_0 + A_1 x + ... + A_n x^n, \quad \text{and} \quad R_n(x) = B_0 + B_1 x + ... + B_n x^n.$$

(6)Find the constants A_i and B_i by plugging y_k into the equation

$$ay'' + by' + cy = g_k(x).$$

Notes

(7)Once all the particular solutions y_k are found, then the particular solution of the original equation is

$$y_p = \sum_{k=1}^{k=N} y_k \ .$$

3.6.3 Method of Undetermined Coefficients

Example: Find a particular solution to the equation

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin(x) - 8e^{-x}.$$

Solution: Let us follow these steps:

(1)First, we notice that the conditions are satisfied to invoke the method of undetermined coefficients.

(2)We split the equation into the following three equations:

$$\begin{array}{ll} (1) & y'' - 3y' - 4y = 3e^{2x} \\ (2) & y'' - 3y' - 4y = 2\sin(x) \\ (3) & y'' - 3y' - 4y = -8e^{-x} \end{array}$$

(3)The root of the characteristic equation $r^2 - 3r - 4 = 0$ are r=-1 and r=4.

(4.1)Particular solution to Equation (1):

Since $\alpha = 2$, and $\beta = 0$, then $\alpha + i\beta = 2$, which is not one of the roots. Then s=0.

The particular solution is given as

$$y_1 = Ae^{2x} \ .$$

If we plug it into the equation (1), we get

$$4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x},$$

which implies A = -1/2, that is,

$$y_1 = -\frac{1}{2}e^{2x} \ .$$

(4.2)Particular solution to Equation (2):

Since $\alpha = 0$, and $\beta = 1$, then $\alpha + i\beta = i$, which is not one of the roots. Then *s*=0.

The particular solution is given as

$$y_2 = A\cos(x) + B\sin(x) .$$

If we plug it into the equation (2), we get

$$(-A\cos(x) - B\sin(x)) - 3(-A\sin(x) + B\cos(x)) - 4(A\cos(x) + B\sin(x)) = 2\sin(x)$$

which implies

$$\begin{cases} -5A - 3B = 0 \\ 3A - 5B = 2 \end{cases}$$

Easy calculations give $A = \frac{3}{17}$, and $B = -\frac{5}{17}$, that is

$$y_2 = \frac{3}{17}\cos(x) - \frac{5}{17}\sin(x) \cdot$$

(4.3)Particular solution to Equation (3):

Since $\alpha = -1$, and $\beta = 0$, then $\alpha + i\beta = -1$ which is one of the roots. Then *s*=1.

The particular solution is given as

$$y_3 = x^1\left(Ae^{-x}\right) .$$

If we plug it into the equation (3), we get

$$A(x-2)e^{-x} - 3A(-x+1)e^{-x} - 4Axe^{-x} = -8e^{x}$$

,

which implies $A = \frac{8}{5}$, that is

$$y_3 = \frac{8}{5}xe^{-x} .$$

(5)A particular solution to the original equation is

$$y_p = -\frac{1}{2}e^{2x} + \frac{3}{17}\cos(x) - \frac{5}{17}\sin(x) + \frac{8}{5}xe^{-x} .$$

Notes

3.6.4 Method of Variation of Parameters

This method has no prior conditions to be satisfied. Therefore, it may sound more general than the previous method. We will see that this method depends on integration while the previous one is purely algebraic which, for some at least, is an advantage.

Consider the equation

$$(NH) \qquad y'' + p(x)y' + q(x)y = g(x).$$

In order to use the method of variation of parameters we need to know that $\{y_1, y_2\}$ is a set of fundamental solutions of the associated homogeneous equation $y'' + p(x)y' + q(x)y = 0$. We know that, in this case, the general solution of the associated homogeneous equation is $y_h = c_1 y_1 + c_2 y_2$. The idea behind the method of variation of parameters is to look for a particular solution such as

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where u_1 and u_2 are functions. From this, the method got its name. The functions u_1 and u_2 are solutions to the system

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(x) \end{cases},$$

which implies

$$\begin{cases} u_1(x) = -\displaystyle\int \frac{y_2(x)g(x)}{W(y_1, y_2)(x)}\,dx, \\ \\ u_2(x) = \displaystyle\int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}\,dx \ . \end{cases},$$

where $W(y_1, y_2)$ is the wronskian of y_1 and y_2. Therefore, we have

$$y_p(x) = -y_1(x)\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}dx + y_2(x)\int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}dx.$$

Summary:Let us summarize the steps to follow in applying this method:

(1)Find $\{y_1, y_2\}$ a set of fundamental solutions of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

(2)Write down the form of the particular solution

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)\ ;$$

(3)Write down the system

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g(x) \end{cases};$$

(4)Solve it. That is, find u_1 and u_2;

(5)Plug u_1 and u_2 into the equation giving the particular solution.

Example: Find the particular solution to

$$y'' + y = 1 + \tan(x); \quad -\frac{\pi}{2} < x < \frac{\pi}{2}\ .$$

Solution: Let us follow the steps:

(1)A set of fundamental solutions of the equation $y'' + y = 0$ is $\{\cos(x), \sin(x)\}$;

(2)The particular solution is given as

$$y_p = u_1 \cos(x) + u_2 \sin(x)\ .$$

(3)We have the system

$$\begin{cases} u_1' \cos(x) + u_2' \sin(x) = 0 \\ -u_1' \sin(x) + u_2' \cos(x) = 1 + \tan(x) \end{cases};$$

(4)We solve for u_1' and u_2', and get

$$u_1'(x) = -\sin(x)\Big(1 + \tan(x)\Big) \text{ and } u_2'(x) = \cos(x)\Big(1 + \tan(x)\Big)\ .$$

Using techniques of integration, we get

$$u_1(x) = \cos(x) + \sin(x) - \ln\left(\sec(x) + \tan(x)\right) \text{ and } u_2(x) = \sin(x)$$
;

(5)The particular solution is:

$$y_p = \cos(x)\Big(\cos(x) + \sin(x) - \ln\left(\sec(x) + \tan(x)\right)\Big) + \sin(x)\Big(\sin(x)$$

or

$$y_p = 1 - \cos(x)\ln\left(\sec(x) + \tan(x)\right).$$

Remark: Note that since the equation is linear, we may still split if necessary. For example, we may split the equation

$$y'' + y = 1 + \tan(x),$$

into the two equations

$$\begin{array}{ll} (1) & y'' + y = 1 \\ (2) & y'' + y = \tan(x) \end{array}$$

then, find the particular solutions y_1 for (1) and y_2 for (2), to generate a particular solution for the original equation by

$$y_p = y_1 + y_2 .$$

There are no restrictions on the method to be used to find y_1 or y_2. For example, we can use the method of undetermined coefficients to find y_1, while for y_2, we are only left with the variation of parameters.

3.7 SUMMARY

In this unit we study An ordinary differential equation of the form

$y'' + P(x)y' + Q(x)y = 0.$
Such an equation has singularities for finite $x = x_0$

We study about nonlinear second order differential equations

$$y'' = f(x, y, y').$$

We study A linear second order differential equations

$$a(x)y'' + b(x)y' + c(x)y = d(x).$$

When d(x) = 0, the equation is called homogeneous, otherwise it is called non-homogeneous.

We learn the homogeneous second order linear equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

3.8 KEYWORD

Homogeneous : of the same kind; alike, denoting a process involving ... Homogenous is a different word, a specialized biological term meaning Non-Homogeneous : made up of different types of people or things : not homogeneous nonhomogeneous neighborhoods the nonhomogenous atmosphere of the planet a nonhomogenous distribution of particles. Variation of Parameters : a method for solving a differential equation by first solving a simpler equation and then generalizing this solution properly so as to satisfy the original equation by treating the arbitrary constants not as constants but as variables.

3.9 EXERCISE

Q. 1 Find the particular solution to

$$y'' + y = 1 + \tan(x); \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Q. 2 Find a particular solution to the equation

$$y'' - 3y' - 4y = 3e^{2x} + 2\sin(x) - 8e^{-x}.$$

Q. 3 Find the general solution of the equation

Notes

$$y'' + (y')^3 y = 0 .$$

Q. 4 Find the particular solution to

$$y'' + y = 1 + \tan(x); \quad -\frac{\pi}{2} < x < \frac{\pi}{2} .$$

3.10 ANSWER TO CHECK IN PROGRESS

Check In Progress-1

Answer Q. 1 Check in section 3

Q. 2 Check in section 4

Check In progress-II

Answer Q. 1 Check in section 5

Q. 2 Check in section 6.1

3.11SUGGESTION READING AND REFERENCES

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UNIT 4 POWER SERIES METOD WITH BESSEL FUNCTION

STRUCTURE

4.0 OBJECTIVE

- In this unit we study about Bessel Differential Equation and its proof
- We also study Bessel function of first kind and Bessel function of second kind
- We study Bessel Function Neumann series
- We study Legendre's Function of first kind and Legendre's Function of second kind
- We study associated Legendre Polynomial

4.1 INTRODUCTION

In this entry the term is used for the cylinder functions of the first kind (which are usually called Bessel functions of the first kind by those authors which use the term Bessel functions for all cylinder functions). For the Bessel functions of the second kind, denoted by Yν (more rarely by Nν) and also called Neumann functions or Weber functions, see Cylinder functions and Neumann function. For the Bessel functions of the third kind see Cylinder functions and Hankel functions.

A function $Z_n(x)$ defined by the recurrence relations

$$Z_{n+1} + Z_{n-1} = \frac{2n}{x} Z_n \tag{1}$$

and

$$Z_{n+1} - Z_{n-1} = -2\frac{dZ_n}{dx}. \tag{2}$$

The Bessel functions are more frequently defined as solutions to the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0. \tag{3}$$

There are two classes of solution, called the Bessel function of the first kind $J_n(x)$ and Bessel function of the second kind $Y_n(x)$. (A Bessel function of the third kind, more commonly called a Hankel function, is a special combination of the first and second kinds.) Several related functions are also defined by slightly modifying the defining equations.

4.2 BESSEL DIFFERENTIAL EQUATION

The Bessel differential equation is the linear second-order ordinary differential equation given by

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + \left(x^2 - n^2\right) y = 0. \quad (1)$$

Equivalently, dividing through by x^2,

$$\frac{d^2 y}{d x^2} + \frac{1}{x} \frac{d y}{d x} + \left(1 - \frac{n^2}{x^2}\right) y = 0. \quad (2)$$

The solutions to this equation define the Bessel functions $J_n(x)$ and $Y_n(x)$
. The equation has a regular singularity at 0 and an
irregular singularity at ∞.

A transformed version of the Bessel differential equation given by Bowman (1958) is

$$x^2 \frac{d^2 y}{d x^2} + (2p + 1) x \frac{d y}{d x} + \left(a^2 x^{2r} + \beta^2\right) y = 0. \quad (3)$$

The solution is

$$y = x^{-p} \left[C_1 J_{q/r}\left(\frac{\alpha}{r} x^r\right) + C_2 Y_{q/r}\left(\frac{\alpha}{r} x^r\right)\right], \quad (4)$$

where

$$q \equiv \sqrt{p^2 - \beta^2}, \quad (5)$$

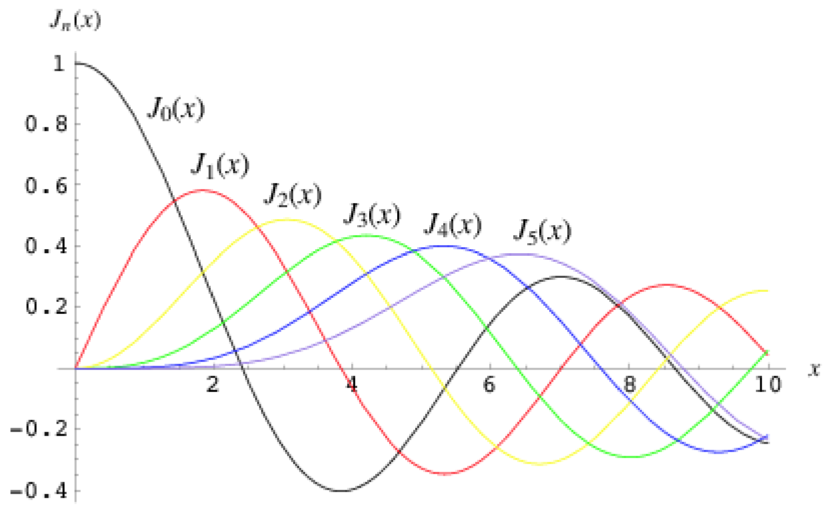
$J_n(x)$ and $Y_n(x)$ are the Bessel functions of the first and second kinds, and C_1 and C_2 are constants. Another form is given by letting $y = x^\alpha J_n(\beta x^\gamma)$, $\eta = y x^{-\alpha}$, and $\xi = \beta x^\gamma$ (Bowman 1958, p. 117), then

$$\frac{d^2 y}{d x^2} - \frac{2 \alpha - 1}{x} \frac{d y}{d x} + \left(\beta^2 \gamma^2 x^{2\gamma-2} + \frac{\alpha^2 - n^2 \gamma^2}{x^2} \right) y = 0. \quad (6)$$

The solution is

$$y = \begin{cases} x^\alpha \left[A J_n(\beta x^\gamma) + B Y_n(\beta x^\gamma)\right] & \text{for integer } n \\ x^\alpha \left[A J_n(\beta x^\gamma) + B J_{-n}(\beta x^\gamma)\right] & \text{for noninteger } n. \end{cases} \quad (7)$$

4.2.1 Bessel Function of the First Kind



The Bessel functions of the first kind $J_n(x)$ are defined as the solutions to the Bessel differential equation

$$x^2 \frac{d^2 y}{d x^2} + x \frac{d y}{d x} + (x^2 - n^2) y = 0 \quad (1)$$

which are nonsingular at the origin. They are sometimes also called cylinder functions or cylindrical harmonics. The above plot shows $J_n(x)$ for $n = 0$, 1, 2, ..., 5. The notation $J_{z,n}$ was first used by Hansen (1843) and subsequently by Schlömilch (1857) to denote what is now written $J_n(2z)$. However, Hansen's definition of the function itself in terms of the generating function

$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z). \quad (2)$$

is the same as the modern one (Watson 1966, p. 14). Bessel used the notation I_k^h to denote what is now called the Bessel function of the first kind (Cajori 1993, vol. 2, p. 279).

The Bessel function $J_n(z)$ can also be defined by the contour integral

$$J_n(z) = \frac{1}{2\pi i} \oint e^{(z/2)(t-1/t)} t^{-n-1} \, dt, \tag{3}$$

where the contour encloses the origin and is traversed in a counterclockwise direction

The Bessel function of the first kind is implemented in the Wolfram Language as BesselJ[*nu*, *z*].

To solve the differential equation, apply Frobenius method using a series solution of the form

$$y = x^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+k}. \tag{4}$$

Plugging into (1) yields

$$x^2 \sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n-2} + x \sum_{n=0}^{\infty} (k+n) a_n x^{k+n-1} + x^2 \sum_{n=0}^{\infty} a_n x^{k+n} - m^2 \sum_{n=0}^{\infty} a_n x^{n+k} = 0 \tag{5}$$

$$\sum_{n=0}^{\infty} (k+n)(k+n-1) a_n x^{k+n} + \sum_{n=0}^{\infty} (k+n) a_n x^{k+n} + \sum_{n=2}^{\infty} a_{n-2} x^{k+n} - m^2 \sum_{n=0}^{\infty} a_n x^{n+k} = 0. \tag{6}$$

The indicial equation, obtained by setting $n = 0$, is

$$a_0 \left[k(k-1) + k - m^2\right] = a_0 \left(k^2 - m^2\right) = 0. \tag{7}$$

Since a_0 is defined as the first nonzero term, $k^2 - m^2 = 0$, so $k = \pm m$. Now, if $k = m$,

$$\sum_{n=0}^{\infty} \left[(m+n)(m+n-1) + (m+n) - m^2\right] a_n x^{m+n} + \sum_{n=2}^{\infty} a_{n-2} x^{m+n} = 0 \tag{8}$$

Notes

$$\sum_{n=0}^{\infty}\left[(m+n)^2 - m^2\right] a_n\, x^{m+n} + \sum_{n=2}^{\infty} a_{n-2}\, x^{m+n} = 0 \tag{9}$$

$$\sum_{n=0}^{\infty} n\,(2\,m+n)\, a_n\, x^{m+n} + \sum_{n=2}^{\infty} a_{n-2}\, x^{m+n} = 0 \tag{10}$$

$$a_1\,(2\,m+1)\, x^{m+1} + \sum_{n=2}^{\infty}\left[a_n\, n\,(2\,m+n) + a_{n-2}\right] x^{m+n} = 0. \tag{11}$$

First, look at the special case $m = -1/2$, then (11) becomes

$$\sum_{n=2}^{\infty}\left[a_n\, n\,(n-1) + a_{n-2}\right] x^{m+n} = 0, \tag{12}$$

so

$$a_n = -\frac{1}{n\,(n-1)}\, a_{n-2}. \tag{13}$$

Now let $n \equiv 2\,l$, where $l = 1$, 2, ...

$$a_{2\,l} = -\frac{1}{2\,l\,(2\,l-1)}\, a_{2\,l-2} \tag{14}$$

$$= \frac{(-1)^l}{[2\,l\,(2\,l-1)]\,[2\,(l-1)\,(2\,l-3)]\cdots[2\cdot 1\cdot 1]}\, a_0 \tag{15}$$

$$= \frac{(-1)^l}{2^l\, l!\,(2\,l-1)!!}\, a_0, \tag{16}$$

which, using the identity $2^l\, l!\,(2\,l-1)!! = (2\,l)!$, gives

$$a_{2\,l} = \frac{(-1)^l}{(2\,l)!}\, a_0. \tag{17}$$

Similarly, letting $n \equiv 2\,l+1$,

$$a_{2\,l+1} = -\frac{1}{(2\,l+1)\,(2\,l)}\, a_{2\,l-1} = \frac{(-1)^l}{[2\,l\,(2\,l+1)]\,[2\,(l-1)\,(2\,l-1)]\cdots[2\cdot 1\cdot 3]\,[1]} \tag{18}$$

which, using the identity $2^l\, l!\,(2\,l+1)!! = (2\,l+1)!$, gives

$$a_{2\,l+1} = \frac{(-1)^l}{2^l\, l!\,(2\,l+1)!!}\, a_1 = \frac{(-1)^l}{(2\,l+1)!}\, a_1. \tag{19}$$

Plugging back into (◇) with $k = m = -1/2$ gives

$$y = x^{-1/2} \sum_{n=0}^{\infty} a_n\, x^n \tag{20}$$

$$= x^{-1/2}\left[\sum_{n=1,3,5,\ldots}^{\infty} a_n\, x^n + \sum_{n=0,2,4,\ldots}^{\infty} a_n\, x^n\right] \tag{21}$$

$$= x^{-1/2}\left[\sum_{l=0}^{\infty} a_{2\,l}\, x^{2\,l} + \sum_{l=0}^{\infty} a_{2\,l+1}\, x^{2\,l+1}\right] \tag{22}$$

$$= x^{-1/2}\left[a_0 \sum_{l=0}^{\infty} \frac{(-1)^l}{(2\,l)!}\, x^{2\,l} + a_1 \sum_{l=0}^{\infty} \frac{(-1)^l}{(2\,l+1)!}\, x^{2\,l+1}\right] \tag{23}$$

$$= x^{-1/2}\,(a_0 \cos x + a_1 \sin x). \tag{24}$$

The Bessel functions of order $\pm 1/2$ are therefore defined as

$$J_{-1/2}(x) \equiv \sqrt{\frac{2}{\pi\, x}}\, \cos x \tag{25}$$

$$J_{1/2}(x) \equiv \sqrt{\frac{2}{\pi\, x}}\, \sin x, \tag{26}$$

so the general solution for $m = \pm 1/2$ is

$$y = a_0'\, J_{-1/2}(x) + a_1'\, J_{1/2}(x). \tag{27}$$

Now, consider a general $m \neq -1/2$. Equation (◇) requires

$$a_1\,(2\,m+1) = 0 \tag{28}$$

$$[a_n\, n\,(2\,m+n) + a_{n-2}]\, x^{m+n} = 0 \tag{29}$$

for $n = 2$, 3, ..., so

$$a_1 = 0 \tag{30}$$

$$a_n = -\frac{1}{n\,(2\,m+n)}\, a_{n-2} \tag{31}$$

for $n = 2$, 3, ... Let $n \equiv 2\,l+1$, where $l = 1$, 2, ..., then

$$a_{2\,l+1} = -\frac{1}{(2\,l+1)\,[2\,(m+l)+1]}\, a_{2\,l-1} \tag{32}$$

$$= \ldots = f(n, m)\, a_1 = 0, \tag{33}$$

where $f(n, m)$ is the function of l and m obtained by iterating the recursion relationship down to a_1. Now let $n \equiv 2\,l$, where $l = 1$, 2, ..., so

$$a_{2\,l} = -\frac{1}{2\,l\,(2\,m+2\,l)}\, a_{2\,l-2} \tag{34}$$

$$= -\frac{1}{4\,l\,(m+l)}\, a_{2\,l-2} \tag{35}$$

$$= \frac{(-1)^l}{[4\,l\,(m+l)]\,[4\,(l-1)\,(m+l-1)]\cdots[4\cdot(m+1)]}\,a_0. \tag{36}$$

Plugging back into (◇),

$$y = \sum_{n=0}^{\infty} a_n\,x^{n+m} = \sum_{n=1,3,5,\ldots}^{\infty} a_n\,x^{n+m} + \sum_{n=0,2,4,\ldots}^{\infty} a_n\,x^{n+m} \tag{37}$$

$$= \sum_{l=0}^{\infty} a_{2\,l+1}\,x^{2\,l+m+1} + \sum_{l=0}^{\infty} a_{2\,l}\,x^{2\,l+m} \tag{38}$$

$$= a_0 \sum_{l=0}^{\infty} \frac{(-1)^l}{[4\,l\,(m+l)]\,[4\,(l-1)\,(m+l-1)]\cdots[4\,(m+1)]}\,x^{2\,l+m} \tag{39}$$

$$= a_0 \sum_{l=0}^{\infty} \frac{\left[(-1)^l\,m\,(m-1)\cdots 1\right]x^{2\,l+m}}{[4\,l\,(m+l)]\,[4\,(l-1)\,(m+l-1)]\cdots[4\,(m+1)\,m\cdots 1]} \tag{40}$$

$$= a_0 \sum_{l=0}^{\infty} \frac{(-1)^l\,m!}{2^{2\,l}\,l!\,(m+l)!}\,x^{2\,l+m}. \tag{41}$$

Now define

$$J_m(x) \equiv \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2\,l+m}\,l!\,(m+l)!}\,x^{2\,l+m}, \tag{42}$$

where the factorials can be generalized to gamma functions for nonintegral m. The above equation then becomes

$$y = a_0\,2^m\,m!\,J_m(x) = a_0'\,J_m(x). \tag{43}$$

Returning to equation (◇) and examining the case $k = -m$,

$$a_1\,(1-2\,m) + \sum_{n=2}^{\infty} [a_n\,n\,(n-2\,m) + a_{n-2}]\,x^{n-m} = 0. \tag{44}$$

However, the sign of m is arbitrary, so the solutions must be the same for $+m$ and $-m$. We are therefore free to replace $-m$ with $-|m|$, so

$$a_1\,(1+2\,|m|) + \sum_{n=2}^{\infty} [a_n\,n\,(n+2\,|m|) + a_{n-2}]\,x^{|m|+n} = 0, \tag{45}$$

and we obtain the same solutions as before, but with m replaced by $|m|$.

$$J_m(x) = \begin{cases} \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l+|m|}\, l!\, (|m|+l)!}\, x^{2l+|m|} & \text{for } |m| \neq \frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \cos x & \text{for } m = -\frac{1}{2} \\ \sqrt{\frac{2}{\pi x}} \sin x & \text{for } m = \frac{1}{2}. \end{cases} \tag{46}$$

We can relate $J_m(x)$ and $J_{-m}(x)$ (when m is an integer) by writing

$$J_{-m}(x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{2^{2l-m}\, l!\, (l-m)!}\, x^{2l-m}. \tag{47}$$

Now let $l \equiv l' + m$. Then

$$J_{-m}(x) = \sum_{l'+m=0}^{\infty} \frac{(-1)^{l'+m}}{2^{2l'+m}\, (l'+m)!\, l!}\, x^{2l'+m} \tag{48}$$

$$= \sum_{l'=-m}^{-1} \frac{(-1)^{l'+m}}{2^{2l'+m}\, l'!\, (l'+m)!}\, x^{2l'+m} + \sum_{l'=0}^{\infty} \frac{(-1)^{l'+m}}{2^{2l'+m}\, l'!\, (l'+m)!}\, x^{2l'+m}. \tag{49}$$

But $l'! = \infty$ for $l' = -m, \ldots, -1$, so the denominator is infinite and the terms on the left are zero. We therefore have

$$J_{-m}(x) = \sum_{l=0}^{\infty} \frac{(-1)^{l+m}}{2^{2l+m}\, l!\, (l+m)!}\, x^{2l+m} \tag{50}$$

$$= (-1)^m J_m(x). \tag{51}$$

Note that the Bessel differential equation is second-order, so there must be two linearly independent solutions. We have found both only for $|m| = 1/2$. For a general nonintegral order, the independent solutions are J_m and J_{-m}. When m is an integer, the general (real) solution is of the form

$$Z_m \equiv C_1 J_m(x) + C_2 Y_m(x), \tag{52}$$

where J_m is a Bessel function of the first kind, Y_m (a.k.a. N_m) is the Bessel function of the second kind (a.k.a. Neumann function or Weber function), and C_1 and C_2 are constants. Complex solutions are given by the Hankel functions (a.k.a. Bessel functions of the third kind).

The Bessel functions are orthogonal in $[0, a]$ according to

$$\int_0^a J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)J_\nu\left(\alpha_{\nu n}\frac{\rho}{a}\right)\rho\,d\rho=\tfrac{1}{2}a^2\left[J_{\nu+1}(\alpha_{\nu m})\right]^2\delta_{mn},\tag{53}$$

where $\alpha_{\nu m}$ is the mth zero of $J\,\nu$ and δ_{mn} is the Kronecker delta (Arfken 1985, p. 592).

Except when $2\,m$ is a negative integer,

$$J_m(z)=\frac{z^{-1/2}}{2^{2m+1/2}\,i^{m+1/2}\,\Gamma(m+1)}M_{0,m}(2\,i\,z),\tag{54}$$

where $\Gamma(x)$ is the gamma function and $M_{0,m}$ is a Whittaker function.

In terms of a confluent hypergeometric function of the first kind, the Bessel function is written

$$J_\nu(z)=\frac{\left(\frac{1}{2}z\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(\nu+1;-\tfrac{1}{4}z^2\right).\tag{55}$$

A derivative identity for expressing higher order Bessel functions in terms of $J_0(z)$ is

$$J_n(z)=i^n\,T_n\left(i\frac{d}{d\,z}\right)J_0(z),\tag{56}$$

where $T_n(z)$ is a Chebyshev polynomial of the first kind. Asymptotic forms for the Bessel functions are

$$J_m(z)\approx\frac{1}{\Gamma(m+1)}\left(\frac{z}{2}\right)^m\tag{57}$$

for $z\ll 1$ and

$$J_m(z)\approx\sqrt{\frac{2}{\pi\,z}}\cos\left(z-\frac{m\,\pi}{2}-\frac{\pi}{4}\right)\tag{58}$$

for $z\gg\left|m^2-1/4\right|$

A derivative identity is $$\frac{d}{d\,x}\left[x^m\,J_m(x)\right]=x^m\,J_{m-1}(x).\tag{59}$$

An integral identity is

$$\int_0^u u' J_0(u')\, du' = u J_1(u). \tag{60}$$

Some sum identities are

$$\sum_{k=-\infty}^{\infty} J_k(x) = 1 \tag{61}$$

(which follows from the generating function (◇) with $t = 1$),

$$1 = [J_0(x)]^2 + 2 \sum_{k=1}^{\infty} [J_k(x)]^2 \tag{62}$$

$$1 = J_0(x) + 2 \sum_{k=1}^{\infty} J_{2k}(x) \tag{63}$$

$$0 = \sum_{k=0}^{2n} (-1)^k J_k(z) J_{2n-k}(z) + 2 \sum_{k=1}^{\infty} J_k(z) J_{2n+k}(z) \tag{64}$$

for $n \geq 1$

$$J_n(2z) = \sum_{k=0}^{n} J_k(z) J_{n-k}(z) + 2 \sum_{k=1}^{\infty} (-1)^k J_k(z) J_{n+k}(z) \tag{65}$$

and the Jacobi-Anger expansion

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta}, \tag{66}$$

which can also be written

$$e^{iz\cos\theta} = J_0(z) + 2 \sum_{n=1}^{\infty} i^n J_n(z) \cos(n\theta). \tag{67}$$

The Bessel function addition theorem states

$$J_n(y+z) = \sum_{m=-\infty}^{\infty} J_m(y) J_{n-m}(z). \tag{68}$$

Various integrals can be expressed in terms of Bessel functions

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(z \sin\theta - n\theta)\, d\theta, \tag{69}$$

which is Bessel's first integral,

Notes

$$J_n(z) = \frac{i^{-n}}{\pi} \int_0^{\pi} e^{i z \cos\theta} \cos(n\theta)\, d\theta \tag{70}$$

$$J_n(z) = \frac{1}{2\pi i^n} \int_0^{2\pi} e^{i z \cos\phi}\, e^{i n \phi}\, d\phi \tag{71}$$

for $n = 1$, 2, ...,

$$J_n(z) = \frac{2}{\pi} \frac{z^n}{(2n-1)!!} \int_0^{\pi/2} \sin^{2n} u \cos(z \cos u)\, du \tag{72}$$

for $n = 1$, 2, ...,

$$J_n(x) = \frac{1}{2\pi i} \int_\gamma e^{(x/2)(z-1/z)}\, z^{-n-1}\, dz \tag{73}$$

for $n > -1/2$. The Bessel functions are normalized so that

$$\int_0^\infty J_n(x)\, dx = 1 \tag{74}$$

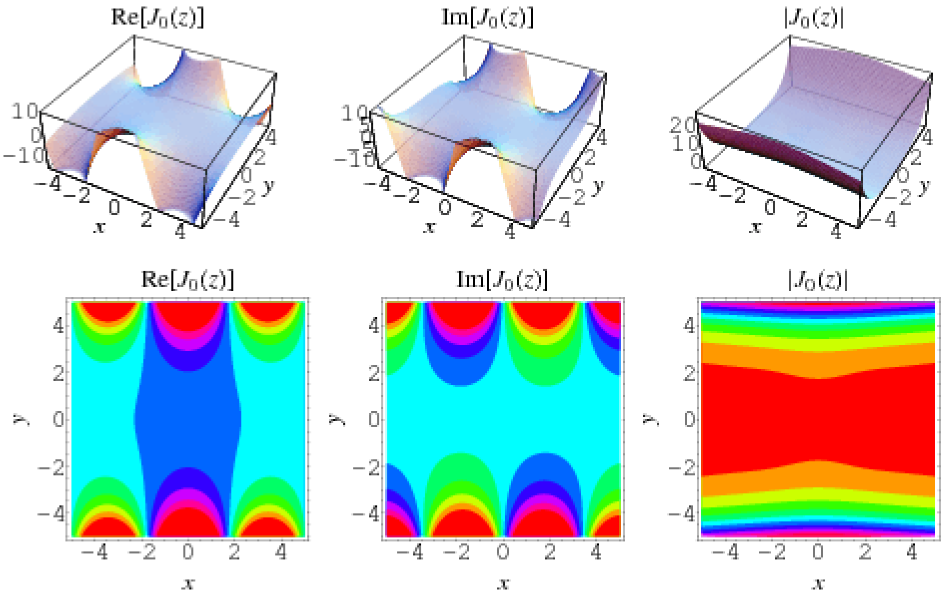
for positive integral (and real) n. Integrals involving $J_1(x)$ include

$$\int_0^\infty \left[\frac{J_1(x)}{x}\right]^2 dx = \frac{4}{3\pi} \tag{75}$$

$$\int_0^\infty \left[\frac{J_1(x)}{x}\right]^2 x\, dx = \frac{1}{2}. \tag{76}$$

Ratios of Bessel functions of the first kind have continued fraction

$$\frac{J_{n-1}(z)}{J_n(z)} = \frac{2n}{z} - \cfrac{\frac{z}{2(n+1)}}{1 - \cfrac{\frac{(z/2)^2}{(n+1)(n+2)}}{1 - \cfrac{\frac{(z/2)^2}{(n+2)(n+3)}}{1 - \ldots}}} \tag{77}$$



The special case of $n = 0$ gives $J_0(z)$ as the series

$$J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4} z^2\right)^k}{(k!)^2} \tag{78}$$

or the integral

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} e^{i z \cos\theta} \, d\theta. \tag{79}$$

Check In Progress-I

Q. 1 Define Bessel differential equation.

Solution : . . . .

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Q. 2 Define Bessel First Order Differential Equation.

Solution : . . . .

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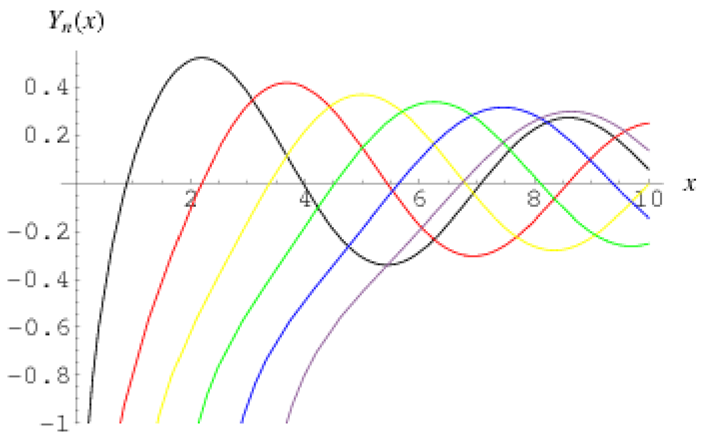
4.2.2 Bessel Function of the Second Kind



A Bessel function of the second kind $Y_n(x)$ sometimes also denoted $N_n(x)$ is a solution to the Bessel differential equation which is singular at the origin. Bessel functions of the second kind are also called Neumann functions or Weber functions. The above plot shows $Y_n(x)$ for $n = 0$, 1, 2, ..., 5. The Bessel function of the second kind is implemented in the Wolfram Language as BesselY[*nu*, *z*].

Let $v \equiv J_m(x)$ be the first solution and u be the other one (since the Bessel differential equation is second-order, there are two linearly independent solutions). Then

$$x u'' + u' + x u = 0 \qquad (1)$$

$$x v'' + v' + x v = 0. \qquad (2)$$

Take $v \times$ (1) minus $u \times$ (2),

$$x (u'' v - u v'') + u' v - u v' = 0 \qquad (3)$$

$$\frac{d}{dx} [x (u' v - u v')] = 0, \qquad (4)$$

so $x (u' v - u v') = B$, where B is a constant. Divide by $x v^2$,

$$\frac{u' v - u v'}{v^2} = \frac{d}{dx} \left(\frac{u}{v} \right) = \frac{B}{x v^2} \qquad (5)$$

$$\frac{u}{v} = A + B \int \frac{dx}{x v^2}. \qquad (6)$$

Rearranging and using $v \equiv J_m(x)$ gives

$$u = A\,J_m(x) + B\,J_m(x)\int \frac{dx}{x\,J_m^2(x)} \tag{7}$$

$$= A'\,J_m(x) + B'\,Y_m(x), \tag{8}$$

where Y_m is the so-called Bessel function of the second kind.

$Y_\nu(z)$ can be defined by

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)} \tag{9}$$

where $J_\nu(z)$ is a Bessel function of the first kind and, for ν an integer n by the series

$$Y_n(z) = -\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi}\ln\left(\frac{1}{2}z\right)J_n(z) - \frac{\left(\frac{1}{2}z\right)^n}{\pi}\sum_{k=0}^{\infty}\left[\psi_0(k+1)+\psi_0(n+k+1)\right]\frac{\left(-\frac{1}{4}z^2\right)^k}{k!\,(n+k)!}, \tag{10}$$

where $\psi_0(x)$ is the digamma function (Abramowitz and Stegun 1972, p. 360).

The function has the integral representations

$$Y_\nu(z) = \frac{1}{\pi}\int_0^\pi \sin(z\sin\theta - \nu\theta)\,d\theta - \frac{1}{\pi}\int_0^\infty\left[e^{\nu t} + e^{-\nu t}(-1)^\nu\right]e^{-z\sinh t}\,dt \tag{11}$$

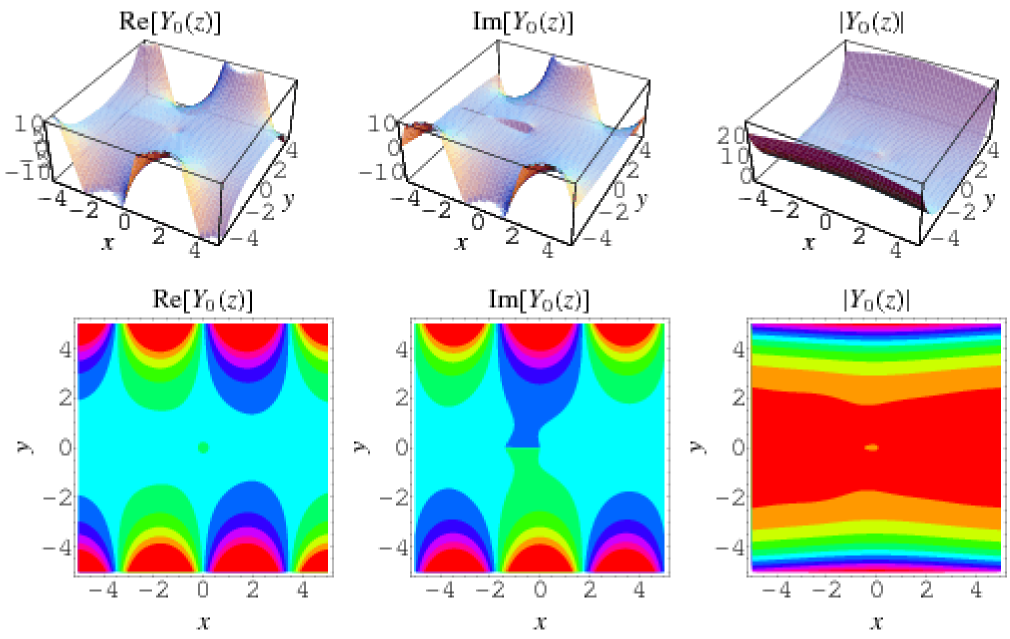
$$= -\frac{2\left(\frac{1}{2}z\right)^{-\nu}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\nu\right)}\int_1^\infty\frac{\cos(z\,t)\,dt}{(t^2-1)^{\nu+1/2}} \tag{12}$$

Asymptotic series are

$$Y_m(x) \sim \begin{cases}\frac{2}{\pi}\left[\ln\left(\frac{1}{2}x\right)+\gamma\right] & m=0,\ x\ll 1\\ -\frac{\Gamma(m)}{\pi}\left(\frac{2}{x}\right)^m & m\neq 0,\ x\ll 1\end{cases} \tag{13}$$

$$Y_m(x) \sim \sqrt{\frac{2}{\pi x}}\,\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad x\gg 1, \tag{14}$$

where $\Gamma(z)$ is a gamma function.



For the special case $n=0$, $Y_0(x)$ is given by the series

$$Y_0(z)=\frac{2}{\pi}\left\{\left[\ln\left(\frac{1}{2}z\right)+\gamma\right]J_0(z)+\sum_{k=1}^{\infty}(-1)^{k+1}H_k\frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2}\right\}, \tag{15}$$

where γ is the Euler-Mascheroni constant and H_n is a harmonic number.

4.2.3 Bessel Function Neumann Series

A series of the form

$$\sum_{n=0}^{\infty}a_n J_{\nu+n}(z), \tag{1}$$

where ν is a real and $J_{\nu+n}(z)$ is a Bessel function of the first kind. Special cases are

$$z^{\nu}=2^{\nu}\,\Gamma\left(\frac{1}{2}\nu+1\right)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^{\nu/2+n}}{n!}J_{\nu/2+n}(z), \tag{2}$$

where $\Gamma(z)$ is the gamma function, and

$$\sum_{n=0}^{\infty}b_n z^{\nu+n}=\sum_{n=0}^{\infty}a_n\left(\frac{1}{2}z\right)^{(\nu+n)/2}J_{(\nu+n)/2}(z), \tag{3}$$

where

$$a_n \equiv \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{2^{\nu+n-2m}\,\Gamma\left(\frac{1}{2}\nu + \frac{1}{2}n - m + 1\right)}{m!} b_{n-2m}, \quad (4)$$

and $\lfloor x \rfloor$ is the floor function.

4.2.4 Bessel Polynomials

Bessel polynomials $\{y_n(x, a, b)\}_{n=0}^{\infty}$ satisfy

$$x^2 y'' + (ax + b)y_n' - n(n + a - 1)y = 0$$

and are given by

$$y_n(x, a, b) = \sum_{k=0}^{n} \frac{n!\,\Gamma(n + k + a - 1)(x/b)^k}{k!(n-k)!\,\Gamma(n + a - 1)}.$$

The ordinary Bessel polynomials are those found with $a = b = 2$, [a2].

The moments associated with the Bessel polynomials satisfy

$$(n + a - 1)\mu_n + b\mu_{n-1} = 0, \qquad n = 0, 1, \ldots,$$

and are given by $\mu_n = (-b)^{n+1} / a(a+1)\ldots(a+n-1)$.

The weight equation is

$$x^2 w' + ((2 - a)x - b)w = N(x),$$

where $N(x)$ is any function with 0 moments. This equation has been solved when

$$N(x) = H(x)e^{-x^{1/4}} \operatorname{sn} x^{1/4},$$

where

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

when $b = 2$ (no restriction), $a - 2 = 2\alpha$ and $\alpha > 6(2/\pi)^4$, [a3]. The weight for the ordinary Bessel polynomials was found by S.S. Kim, K.H. Kwon and S.S. Han, [a1], after over 40 years of search.

Using the three-term recurrence relation

$$(n + a - 1)(2n + a - 2)y_{n+1}(x, a, b) =$$
$$= \left[(2n + a)(2n + a - 2)\left(\frac{x}{b}\right) + (a - 2)\right].$$

Notes

$.(2n+\alpha-1)y_n+n(2n+\alpha)y_{n-1}$,

the norm square $\int_0^\infty g_n(x,\alpha,b)^2 w(x)\,dx$ is easily calculated and equals $(-b)^{k+1}k^{(n)}/(k+\alpha+n_1)^{(k+n)}$, [a2],

where $x^{(k)}=x(x-1)\ldots(x-k+1)$. Clearly, w generates a Krein space on $[0,\infty)$.

4.3 EULER-CAUCHY EQUATIONS

An Euler-Cauchy equation is

$$(EC) \qquad x^2y''+bxy'+cy=0$$

where *b* and *c* are constant numbers. Let us consider the change of variable

$$x=e^t.$$

Then we have

$$\frac{dy}{dx}=e^{-t}\frac{dy}{dt}, \quad \text{and} \quad \frac{d^2y}{dx^2}=\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)e^{-2t}.$$

The equation (*EC*) reduces to the new equation

$$\frac{d^2y}{dt^2}+(b-1)\frac{dy}{dt}+cy=0\,.$$

We recognize a second order differential equation with constant coefficients. Therefore, we use the previous sections to solve it. We summarize below all the cases:

(1)Write down the characteristic equation

$$r^2+(b-1)r+c=0\,.$$

(2)If the roots r_1 and r_2 are distinct real numbers, then the general solution of (*EC*) is given by

$$y(x)=c_1\,|x|^{r_1}+c_2\,|x|^{r_2}.$$

(3)If the roots r_1 and r_2 are equal ($r_1=r_2$), then the general solution of (*EC*) is

$$y(x)=\left(c_1+c_2\ln|x|\right)|x|^{r_1}.$$

(4)If the roots r_1 and r_2 are complex numbers, then the general solution of (*EC*) is

$$y = \Big(c_1 \cos(\beta \ln|x|) + c_2 \sin(\beta \ln|x|)\Big)|x|^\alpha$$

where $\alpha = -(b-1)/2$ and $\beta = \sqrt{4c-(b-1)^2}/2$.

Example : Find the general solution to

$$x^2y'' - xy' + y = 0\ .$$

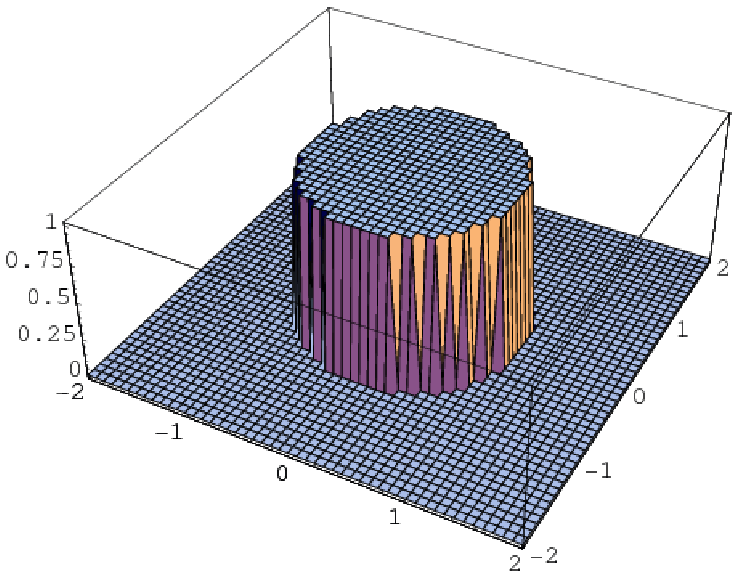
Solution: First we recognize that the equation is an Euler-Cauchy equation, with b=-1 and c=1.

1Characteristic equation is r^2 -2r + 1=0.

2Since 1 is a double root, the general solution is

$$y(x) = \Big(c_1 + c_2 \ln|x|\Big)|x|.$$

4.4 CYLINDER FUNCTION



The cylinder function is defined as

$$C(x, y) \equiv \begin{cases} 1 & \text{for } \sqrt{x^2 + y^2} \leq a \\ 0 & \text{for } \sqrt{x^2 + y^2} > a. \end{cases} \quad (1)$$

The Bessel functions are sometimes also called cylinder functions.

To find the Fourier transform of the cylinder function, let

$$k_x = k \cos\alpha \quad (2)$$

$$k_y = k \sin\alpha \quad (3)$$

and

$$x = r\cos\theta \quad (4)$$

$$y = r\sin\theta. \quad (5)$$

Then

$$F(k, a) = \mathcal{F}_{x\,y}[C(x, y)](k, a) \quad (6)$$

$$= \int_0^{2\pi} \int_0^a e^{i(k\,r\cos\alpha\cos\theta + k\,r\sin\alpha\sin\theta)}\, r\,d\,r\,d\,\theta \quad (7)$$

$$= \int_0^{2\pi} \int_0^a e^{i\,k\,r\cos(\theta-\alpha)}\, r\,d\,r\,d\,\theta. \quad (8)$$

Let $b = \theta - \alpha$, so $d\,b = d\,\theta$. Then

$$F(k, a) = \int_{-\alpha}^{2\pi-\alpha} \int_0^a e^{i\,k\,r\cos b}\, r\,d\,r\,d\,b \quad (9)$$

$$= \int_0^{2\pi} \int_0^a e^{i\,k\,r\cos b}\, r\,d\,r\,d\,b \quad (10)$$

$$= 2\pi \int_0^a J_0(k\,r)\, r\,d\,r \quad (11)$$

$$= \frac{2\pi a}{k} J_1(k\,a) \quad (12)$$

$$= 2\pi a^2 \frac{J_1(k\,a)}{k\,a}. \quad (13)$$

where $J_n(x)$ is a Bessel function of the first kind.

As defined by Watson (1966), a "cylinder function" is any function which satisfies the recurrence relations

$$C_{\nu-1}(z) + C_{\nu+1}(z) = \frac{2\nu}{z} C_\nu(z) \quad (14)$$

$$C_{\nu-1}(z) - C_{\nu+1}(z) = 2\,C_\nu'(z). \quad (15)$$

This class of functions can be expressed in terms of Bessel functions.

4.4.1 Hemispherical Function

The hemisphere function is defined as

$$H(x,\ y) = \begin{cases} \sqrt{a - x^2 - y^2} & \text{for } \sqrt{x^2 + y^2} \le a \\ 0 & \text{for } \sqrt{x^2 + y^2} > a. \end{cases} \qquad (1)$$

Watson (1966) defines a hemispherical function as a function S which satisfies the recurrence relations

$$S_{n-1}(z) - S_{n+1}(z) = 2\, S_n'(z) \qquad (2)$$

with

$$S_1(z) = -S_0'(z).$$

(3)

4.5 LEGENDRE DIFFERENTIAL EQUATION

The Legendre differential equation is the second-order ordinary differential equation

$$\left(1 - x^2\right) \frac{d^2 y}{d x^2} - 2\, x \frac{d y}{d x} + l\,(l + 1)\, y = 0, \qquad (1)$$

which can be rewritten

$$\frac{d}{d x} \left[\left(1 - x^2\right) \frac{d y}{d x}\right] + l\,(l + 1)\, y = 0. \qquad (2)$$

The above form is a special case of the so-called "associated Legendre differential equation" corresponding to the case $m = 0$. The Legendre differential equation has regular singular points at -1, 1, and ∞.

If the variable x is replaced by $\cos\theta$, then the Legendre differential equation becomes

$$\frac{d^2 y}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta} + l(l+1)y = 0, \tag{3}$$

derived below for the associated ($m \neq 0$) case.

Since the Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution $P_l(x)$ which is regular at finite points is called a Legendre function of the first kind, while a solution $Q_l(x)$ which is singular at ± 1 is called a Legendre function of the second kind. If l is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial.

The Legendre differential equation can be solved using the Frobenius method by making a series expansion with $k = 0$,

$$y = \sum_{n=0}^{\infty} a_n x^n \tag{4}$$

$$y' = \sum_{n=0}^{\infty} n\, a_n x^{n-1} \tag{5}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2}. \tag{6}$$

Plugging in,

$$(1-x^2)\sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n\, a_n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0 \tag{7}$$

$$\sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)\, a_n x^n \tag{8}$$

$$-2x\sum_{n=0}^{\infty} n\, a_n x^{n-1} + l(l+1)\sum_{n=0}^{\infty} a_n x^n = 0 \tag{9}$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)\, a_n x^n \tag{10}$$

$$-2\sum_{n=0}^{\infty} n\, a_n\, x^n + l\,(l+1)\sum_{n=0}^{\infty} a_n\, x^n = 0 \tag{11}$$

$$\sum_{n=0}^{\infty}(n+2)\,(n+1)\,a_{n+2}\,x^n - \sum_{n=0}^{\infty} n\,(n-1)\,a_n\,x^n \tag{12}$$

$$-2\sum_{n=0}^{\infty} n\, a_n\, x^n + l\,(l+1)\sum_{n=0}^{\infty} a_n\, x^n = 0 \tag{13}$$

$$\sum_{n=0}^{\infty}\{(n+1)\,(n+2)\,a_{n+2} + [-n\,(n-1) - 2\,n + l\,(l+1)]\,a_n\} = 0, \tag{14}$$

so each term must vanish and

$$(n+1)\,(n+2)\,a_{n+2} + [-n\,(n+1) + l\,(l+1)]\,a_n = 0 \tag{15}$$

$$a_{n+2} = \frac{n\,(n+1) - l\,(l+1)}{(n+1)\,(n+2)}\,a_n \tag{16}$$

$$= -\frac{[l + (n+1)]\,(l-n)}{(n+1)\,(n+2)}\,a_n. \tag{17}$$

Therefore,

$$a_2 = -\frac{l\,(l+1)}{1\cdot 2}\,a_0 \tag{18}$$

$$a_4 = -\frac{(l-2)\,(l+3)}{3\cdot 4}\,a_2 \tag{19}$$

$$= (-1)^2\,\frac{[(l-2)\,l]\,[(l+1)\,(l+3)]}{1\cdot 2\cdot 3\cdot 4}\,a_0 \tag{20}$$

$$a_6 = -\frac{(l-4)\,(l+5)}{5\cdot 6}\,a_4 \tag{21}$$

$$= (-1)^3\,\frac{[(l-4)\,(l-2)\,l]\,[(l+1)\,(l+3)\,(l+5)]}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\,a_0, \tag{22}$$

so the even solution is

$$y_1\,(x) = 1 + \sum_{n=1}^{\infty}(-1)^n\,\frac{[(l-2\,n+2)\cdots(l-2)\,l]\,[(l+1)\,(l+3)\cdots(l+2\,n-1)]}{(2\,n)!}\,x \tag{23}$$

Similarly, the odd solution is

$$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{[(l-2n+1)\cdots(l-3)(l-1)][(l+2)(l+4)\cdots(l+2n)]}{(2n+1)!} \qquad (24)$$

If l is an even integer, the series $y_1(x)$ reduces to a polynomial of degree l with only even powers of x and the series $y_2(x)$ diverges. If l is an odd integer, the series $y_2(x)$ reduces to a polynomial of degree l with only odd powers of x and the series $y_1(x)$ diverges. The general solution for an integer l is then given by the Legendre polynomials

$$P_n(x) = c_n \begin{cases} y_1(x) & \text{for } l \text{ even} \\ y_2(x) & \text{for } l \text{ odd} \end{cases} \qquad (25)$$

$$= c_n \begin{cases} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}(l+1); \frac{1}{2}, x^2\right) & \text{for } l \text{ even} \\ x\,{}_2F_1\left(\frac{1}{2}(l+2), \frac{1}{2}(1-l); \frac{3}{2}; x^2\right) & \text{for } l \text{ odd} \end{cases} \qquad (26)$$

where c_n is chosen so as to yield the normalization $P_n(1) = 1$ and ${}_2F_1(a, b; c; z)$ is a hypergeometric function.

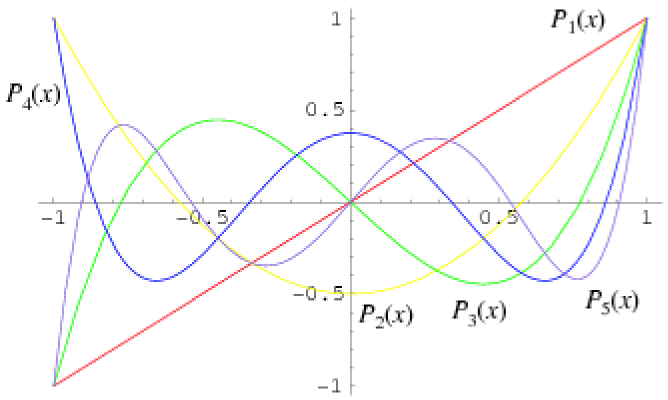
A generalization of the Legendre differential equation is known as the associated Legendre differential equation.

Moon and Spencer call the differential equation

$$(1-x^2)y'' - 2xy' - \left[k^2 a^2 (x^2-1) - p(p+1) - \frac{q^2}{x^2-1}\right] y = 0 \qquad (27)$$

the Legendre wave function equation

4.5.1 Legendre Polynomial



The Legendre polynomials, sometimes called Legendre functions of the first kind, Legendre coefficients, or zonal harmonics (Whittaker and Watson 1990, p. 302), are solutions to the Legendre differential equation. If l is an integer, they are polynomials. The Legendre polynomials $P_n(x)$ are illustrated above for $x \in [-1, 1]$ and $n = 1$, 2, ..., 5. They are implemented in the Wolfram Language as LegendreP[*n*, *x*].

The associated Legendre polynomials $P_l^m(x)$ and P_l^{-m} are solutions to the associated Legendre differential equation, where l is a positive integer and $m = 0$, ..., l.

The Legendre polynomial $P_n(z)$ can be defined by the contour integral

$$P_n(z) = \frac{1}{2\pi i} \oint (1 - 2tz + t^2)^{-1/2} t^{-n-1} \, dt, \tag{1}$$

where the contour encloses the origin and is traversed in a counterclockwise direction

The first few Legendre polynomials are

$$P_0(x) = 1 \tag{2}$$

$$P_1(x) = x \tag{3}$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \tag{4}$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \tag{5}$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \tag{6}$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right) \tag{7}$$

$$P_6(x) = \frac{1}{16}\left(231x^6 - 315x^4 + 105x^2 - 5\right). \tag{8}$$

When ordered from smallest to largest powers and with the denominators factored out, the triangle of nonzero coefficients is 1, 1, -1, 3, -3, 5, 3, -30, ... (OEIS A008316). The leading denominators are 1, 1, 2, 2, 8, 8, 16, 16, 128, 128, 256, 256, ... (OEIS A060818).

The first few powers in terms of Legendre polynomials are

$$x = P_1(x) \tag{9}$$

$$x^2 = \frac{1}{3}\left[P_0(x) + 2P_2(x)\right] \tag{10}$$

$$x^3 = \frac{1}{5}\left[3P_1(x) + 2P_3(x)\right] \tag{11}$$

$$x^4 = \frac{1}{35}\left[7P_0(x) + 20P_2(x) + 8P_4(x)\right] \tag{12}$$

$$x^5 = \frac{1}{63}\left[27P_1(x) + 28P_3(x) + 8P_5(x)\right] \tag{13}$$

$$x^6 = \frac{1}{231}\left[33P_0(x) + 110P_2(x) + 72P_4(x) + 16P_6(x)\right] \tag{14}$$

A closed form for these is given by

$$x^n = \sum_{l=n,n-2,\ldots} \frac{(2l+1)\,n!}{2^{(n-l)/2}\left(\frac{1}{2}(n-l)\right)!\,(l+n+1)!!} P_l(x) \tag{15}$$

For Legendre polynomials and powers up to exponent 12,

The Legendre polynomials can also be generated using Gram-Schmidt orthonormalization in the open interval $(-1, 1)$ with the weighting function 1.

$$P_0(x) = 1 \tag{16}$$

$$P_1(x) = \left[x - \frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} \right] \cdot 1 \quad (17)$$

$$= x \quad (18)$$

$$P_2(x) = x\left[x - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} \right] - \left[\frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} \right] \cdot 1 \quad (19)$$

$$= x^2 - \tfrac{1}{3} \quad (20)$$

$$P_3(x) = \left[x - \frac{\int_{-1}^{1} x\left(x^2 - \frac{1}{3}\right)^2 dx}{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx} \right] \left(x^2 - \tfrac{1}{3}\right) - \left[\frac{\int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx}{\int_{-1}^{1} x^2\,dx} \right] x \quad (21)$$

$$= x^3 - \tfrac{3}{5}x. \quad (22)$$

Normalizing so that $P_n(1) = 1$ gives the expected Legendre polynomials.

The "shifted" Legendre polynomials are a set of functions analogous to the Legendre polynomials, but defined on the interval (0, 1). They obey the orthogonality relationship

$$\int_0^1 \overline{P}_m(x)\,\overline{P}_n(x)\,dx = \frac{1}{2n+1}\delta_{mn}. \quad (23)$$

The first few are

$$\overline{P}_0(x) = 1 \quad (24)$$

$$\overline{P}_1(x) = 2x - 1 \quad (25)$$

$$\overline{P}_2(x) = 6x^2 - 6x + 1 \quad (26)$$

$$\overline{P}_3(x) = 20x^3 - 30x^2 + 12x - 1. \quad (27)$$

The Legendre polynomials are orthogonal over $(-1, 1)$ with weighting function 1 and satisfy

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}, \quad (28)$$

where δ_{mn} is the Kronecker delta.

Notes

The Legendre polynomials are a special case of the Gegenbauer polynomials with $\alpha = 1/2$, a special case of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha = \beta = 0$, and can be written as a hypergeometric function using Murphy's formula

$$P_n(x) = P_n^{(0,0)}(x) = {}_2F_1\left(-n, n+1; 1; \tfrac{1}{2}(1-x)\right) \tag{29}$$

The Rodrigues representation provides the formula

$$P_l(x) = \frac{1}{2^l\, l!} \frac{d^l}{d\,x^l} \left(x^2 - 1\right)^l, \tag{30}$$

which yields upon expansion

$$P_l(x) = \frac{1}{2^l} \sum_{k=0}^{\lfloor l/2 \rfloor} \frac{(-1)^k (2l - 2k)!}{k!\,(l-k)!\,(l-2k)!} x^{l-2k} \tag{31}$$

$$= \frac{1}{2^l} \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \binom{l}{k} \binom{2l-2k}{l} x^{l-2k} \tag{32}$$

where $\lfloor r \rfloor$ is the floor function. Additional sum formulas include

$$P_l(x) = \frac{1}{2^l} \sum_{k=0}^{l} \binom{l}{k}^2 (x-1)^{l-k} (x+1)^k \tag{33}$$

$$= \sum_{k=0}^{l} \binom{l}{k} \binom{-l-1}{k} \left(\frac{1-x}{2}\right)^k \tag{34}$$

In terms of hypergeometric functions, these can be written

$$P_n(x) = \left(\frac{x-1}{2}\right)^n {}_2F_1\left(-n, -n; 1; (x+1)/(x-1)\right) \tag{35}$$

$$P_n(x) = \binom{2n}{n} \frac{x^n}{2^n} {}_2F_1\left(-n/2, (1-n)/2; 1/2 - n; x^{-2}\right) \tag{36}$$

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; (1-x)/2\right) \tag{37}$$

A generating function for $P_n(x)$ is given by

$$g(t,x) = \left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n. \tag{38}$$

Take $\partial g / \partial t$,

$$-\tfrac{1}{2}\left(1 - 2xt + t^2\right)^{-3/2}(-2x + 2t) = \sum_{n=0}^{\infty} n\, P_n(x)\, t^{n-1}. \tag{39}$$

Multiply (39) by $2t$,

$$-t\left(1 - 2xt + t^2\right)^{-3/2}(-2x + 2t) = \sum_{n=0}^{\infty} 2n\, P_n(x)\, t^n \tag{40}$$

and add (38) and (40),

$$\left(1 - 2xt + t^2\right)^{-3/2}\left[\left(2xt - 2t^2\right) + \left(1 - 2xt + t^2\right)\right] = \sum_{n=0}^{\infty} (2n + 1)\, P_n(x)\, t^n \tag{41}$$

This expansion is useful in some physical problems, including expanding the Heyney-Greenstein phase function and computing the charge distribution on a sphere. Another generating function is given by

$$\sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n = e^{xz} J_0\left(z\sqrt{1 - x^2}\right), \tag{42}$$

where $J_0(x)$ is a zeroth order Bessel function of the first kind.

The Legendre polynomials satisfy the recurrence relation

$$(l + 1)\, P_{l+1}(x) - (2l + 1)\, x\, P_l(x) + l\, P_{l-1}(x) = 0 \tag{43}$$

In addition,

$$\left(1 - x^2\right) P_n'(x) = -n\, x\, P_n(x) + n\, P_{n-1}(x) = (n + 1)\, x\, P_n(x) - (n + 1)\, P_{n+1}(x) \tag{44}$$

A complex generating function is

$$P_l(x) = \frac{1}{2\pi i} \int \left(1 - 2zx + z^2\right)^{-1/2} z^{-l-1}\, dz, \tag{45}$$

and the Schläfli integral is

$$P_l(x) = \frac{(-1)^l}{2^l} \frac{1}{2\pi i} \int \frac{\left(1-z^2\right)^l}{(z-x)^{l+1}} \, dz. \tag{46}$$

Integrals over the interval $[x, 1]$ include the general formula

$$\int_x^1 P_m(x) \, dx = \frac{\left(1-x^2\right)}{m(m+1)} \frac{dP_m(x)}{dx} \tag{47}$$

for $m \neq 0$ from which the special case

$$\int_0^1 P_m(x) \, dx = \frac{P_{m-1}(0) - P_{m+1}(0)}{2m+1} \tag{48}$$

$$= \begin{cases} 1 & m = 0 \\ 0 & m \text{ even } \neq 0 \\ (-1)^{(m-1)/2} \dfrac{m!!}{m(m+1)(m-1)!!} & m \text{ odd} \end{cases} \tag{49}$$

Follows. For the integral over a product of Legendre functions,

$$\int_x^1 P_m(x) P_n(x) \, dx = \frac{\left(1-x^2\right)\left[P_n(x) P_m'(x) - P_m(x) P_n'(x)\right]}{m(m+1) - n(n+1)} \tag{50}$$

for $m \neq n$ which gives the special case

$$\int_0^1 P_m(x) P_n(x) \, dx = \begin{cases} \dfrac{1}{2n+1} & m = n \\ 0 & m \neq n, m, n \text{ both even or odd} \\ f_{m,n} & m \text{ even}, n \text{ odd} \\ f_{n,m} & m \text{ odd}, n \text{ even} \end{cases} \tag{51}$$

where

$$f_{m,n} \equiv \frac{(-1)^{(m+n+1)/2} m! \, n!}{2^{m+n-1} (m-n)(m+n+1) \left[\left(\frac{1}{2} m\right)!\right]^2 \left\{\left[\frac{1}{2}(n-1)\right]!\right\}^2} \tag{52}$$

The latter is a special case of

$$\int_0^1 P_\mu(x) P_\nu(x)\, dx = \frac{A \sin\left(\frac{1}{2}\pi\nu\right)\cos\left(\frac{1}{2}\pi\mu\right) - A^{-1}\sin\left(\frac{1}{2}\pi\mu\right)\cos\left(\frac{1}{2}\pi\nu\right)}{\frac{1}{2}\pi(\nu-\mu)(\mu+\nu+1)}, \tag{53}$$

where

$$A \equiv \frac{\Gamma\left(\frac{1}{2}(\mu+1)\right)\Gamma\left(1+\frac{1}{2}\nu\right)}{\Gamma\left(\frac{1}{2}(\nu+1)\right)\Gamma\left(1+\frac{1}{2}\mu\right)} \tag{54}$$

and $\Gamma(z)$ is a gamma function

Integrals over $[-1, 1]$ with weighting functions x and x^2 are given by

$$\int_{-1}^1 x\, P_L(x) P_N(x)\, dx = \begin{cases} \dfrac{2(L+1)}{(2L+1)(2L+3)} & \text{for } N = L+1 \\ \dfrac{2L}{(2L-1)(2L+1)} & \text{for } N = L-1 \end{cases} \tag{55}$$

$$\int_{-1}^1 x^2 P_L(x) P_N(x)\, dx = \begin{cases} \dfrac{2(L+1)(L+2)}{(2L+1)(2L+3)(2L+5)} & \text{for } N = L+2 \\ \dfrac{2\left(2L^2+2L-1\right)}{(2L-1)(2L+1)(2L+3)} & \text{for } N = L \\ \dfrac{2L(L-1)}{(2L-3)(2L-1)(2L+1)} & \text{for } N = L-2 \end{cases} \tag{56}$$

The Laplace transform is given by

$$\mathcal{L}[P_n(t)](s) =$$

$$\begin{cases} \frac{1}{2}\sqrt{\pi}\left[\sqrt{\frac{2}{s}}\, I_{-n-1/2}(s) - \frac{1}{2} s\, {}_1F_2\left(1; 2+\frac{1}{2}n, \frac{1}{2}(3-n); \frac{1}{4}s^2\right)\right] & \text{for } n \text{ even} \\ \frac{1}{2}\sqrt{\pi}\left[\sqrt{\frac{2}{s}}\, I_{-n-1/2}(s) + {}_1F_2\left(1; \frac{1}{2}(3+n), 1-\frac{1}{2}n; \frac{1}{4}s^2\right)\right] & \text{for } n \text{ odd,} \end{cases} \tag{57}$$

where $I_n(s)$ is a modified Bessel function of the first kind.

A sum identity is given by

$$1 - [P_n(x)]^2 = \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2}\left[\frac{P_n(x)}{P_n'(x_\nu)(x-x_\nu)}\right]^2, \tag{58}$$

where x_ν is the νth root of $P_n(x)$ (Szegö 1975, p. 348). A similar identity is

$$\sum_{\nu=1}^{n} \frac{1-x_\nu^2}{(n+1)^2\,[P_{n+1}(x_\nu)]^2} = 1, \tag{59}$$

which is responsible for the fact that the sum of weights in Legendre-Gauss quadrature is always equal to 2.

Check In Progress-II

Q. 1 Define Legendre's differential equation.

Solution : . . . .

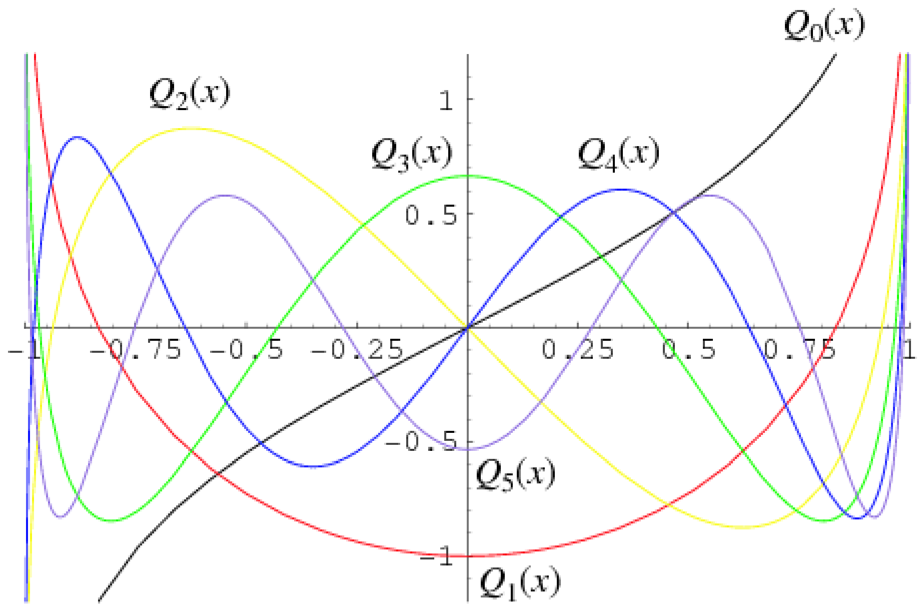
Q. 2 Define Legendre's Polynomial .

Solution : . . . .

4.5.2 Legendre Function of the First Kind

The (associated) Legendre function of the first kind $P_n^m(z)$ is the solution to the Legendre differential equation which is regular at the origin. For m, n integers and z real, the Legendre function of the first kind simplifies to a polynomial, called the Legendre polynomial. The associated Legendre function of first kind is given by the Wolfram Language command LegendreP[*n*, *m*, *z*], and the unassociated function by LegendreP[*n*, *z*].

4.5.3 Legendre Function of the Second Kind



The second solution $Q_l(x)$ to the Legendre differential equation. The Legendre functions of the second kind satisfy the same recurrence relation as the Legendre polynomials. The Legendre functions of the second kind are implemented in the Wolfram

Language as LegendreQ[*l*, *x*]. The first few are

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{1}$$

$$Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1 \tag{2}$$

$$Q_2(x) = \frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{3x}{2} \tag{3}$$

$$Q_3(x) = \frac{5x^3-3x}{4}\ln\left(\frac{1+x}{1-x}\right) - \frac{5x^2}{2} + \frac{2}{3}. \tag{4}$$

The associated Legendre functions of the second kind $Q_l^m(x)$ are the second solution to the associated Legendre differential equation, and are implemented in the Wolfram

Language as LegendreQ[*l*, *m*, *x*] $Q_\nu^\mu(x)$ has derivative about 0 of

$$\left[\frac{dQ_\nu^\mu(x)}{dx}\right]_{x=0} = \frac{2^\mu\sqrt{\pi}\cos\left[\frac{1}{2}\pi(\nu+\mu)\right]\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+1\right)}{\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}\right)} \tag{5}$$

The logarithmic derivative is

$$\left[\frac{d \ln Q_\lambda^\mu(z)}{d z}\right]_{z=0} = 2 \exp\left\{\tfrac{1}{2} \pi i \operatorname{sgn}(\mathrm{I}[z])\right\} \frac{\left[\frac{1}{2}(\lambda+\mu)\right]! \left[\frac{1}{2}(\lambda-\mu)\right]!}{\left[\frac{1}{2}(\lambda+\mu-1)\right]! \left[\frac{1}{2}(\lambda-\mu-1)\right]!} \qquad (6)$$

4.5.4 Associated Legendre Polynomial

The associated Legendre polynomials $P_l^m(x)$ and $P_l^{-m}(x)$ generalize the Legendre polynomials $P_l(x)$ and are solutions to the associated Legendre differential equation, where l is a positive integer and $m = 0$, ..., l. They are implemented in the Wolfram Language as LegendreP[*l*, *m*, *x*]. For positive m, they can be given in terms of the unassociated polynomials by

$$P_l^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{d^m}{d x^m} P_l(x) \qquad (1)$$

$$= \frac{(-1)^m}{2^l\, l!} \left(1-x^2\right)^{m/2} \frac{d^{l+m}}{d x^{l+m}} \left(x^2-1\right)^l, \qquad (2)$$

where $P_l(x)$ are the unassociated Legendre polynomials. The associated Legendre polynomials for negative m are then defined by

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x). \qquad (3)$$

There are two sign conventions for associated Legendre polynomials. Some authors (e.g., Arfken 1985, pp. 668-669) omit the Condon-Shortley phase $(-1)^m$, while others include it (e.g., Abramowitz and Stegun 1972, Press *et al.* 1992, and the LegendreP[*l*, *m*, *z*] command in the Wolfram Language). Care is therefore needed in comparing polynomials obtained from different sources. One possible way to distinguish the two conventions is due to Abramowitz and Stegun (1972, p. 332), who use the notation

$$P_{l m}(x) \equiv (-1)^m P_l^m(x) \qquad (4)$$

to distinguish the two.

Associated polynomials are sometimes called Ferrers' functions (Sansone 1991, p. 246). If $m = 0$, they reduce to the unassociated polynomials. The associated Legendre functions are part of the spherical harmonics, which are the solution of Laplace's equation in spherical coordinates. They are orthogonal over $[-1, 1]$ with the weighting function 1

$$\int_{-1}^{1} P_l^m(x) P_{l'}^m(x)\, dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}, \tag{5}$$

and orthogonal over $[-1, 1]$ with respect to m with the weighting function $(1-x^2)^{-1}$,

$$\int_{-1}^{1} P_l^m(x) P_l^{m'}(x) \frac{dx}{1-x^2} = \frac{(l+m)!}{m(l-m)!} \delta_{mm'}. \tag{6}$$

The associated Legendre polynomials also obey the following recurrence relations

$$(l-m) P_l^m(x) = x(2l-1) P_{l-1}^m(x) - (l+m-1) P_{l-2}^m(x). \tag{7}$$

Letting $x \equiv \cos\theta$ (commonly denoted μ in this context),

$$\frac{d P_l^m(\mu)}{d\theta} = \frac{l \mu P_l^m(\mu) - (l+m) P_{l-1}^m(\mu)}{\sqrt{1-\mu^2}} \tag{8}$$

$$(2l+1) \mu P_l^m(\mu) = (l+m) P_{l-1}^m(\mu) + (l-m+1) P_{l+1}^m(\mu). \tag{9}$$

Additional identities are

$$P_l^l(x) = (-1)^l (2l-1)!! (1-x^2)^{l/2} \tag{10}$$

$$P_{l+1}^l(x) = x(2l+1) P_l^l(x). \tag{11}$$

Including the factor of $(-1)^m$, the first few associated Legendre polynomials are

Notes

$$P_0^0(x) = 1 \tag{12}$$

$$P_1^0(x) = x \tag{13}$$

$$P_1^1(x) = -\left(1 - x^2\right)^{1/2} \tag{14}$$

$$P_2^0(x) = \tfrac{1}{2}\left(3x^2 - 1\right) \tag{15}$$

$$P_2^1(x) = -3x\left(1 - x^2\right)^{1/2} \tag{16}$$

$$P_2^2(x) = 3\left(1 - x^2\right) \tag{17}$$

$$P_3^0(x) = \tfrac{1}{2}x\left(5x^2 - 3\right) \tag{18}$$

$$P_3^1(x) = \tfrac{3}{2}\left(1 - 5x^2\right)\left(1 - x^2\right)^{1/2} \tag{19}$$

$$P_3^2(x) = 15x\left(1 - x^2\right) \tag{20}$$

$$P_3^3(x) = -15\left(1 - x^2\right)^{3/2} \tag{21}$$

$$P_4^0(x) = \tfrac{1}{8}\left(35x^4 - 30x^2 + 3\right) \tag{22}$$

$$P_4^1(x) = \tfrac{5}{2}x\left(3 - 7x^2\right)\left(1 - x^2\right)^{1/2} \tag{23}$$

$$P_4^2(x) = \tfrac{15}{2}\left(7x^2 - 1\right)\left(1 - x^2\right) \tag{24}$$

$$P_4^3(x) = -105x\left(1 - x^2\right)^{3/2} \tag{25}$$

$$P_4^4(x) = 105\left(1 - x^2\right)^2 \tag{26}$$

$$P_5^0(x) = \tfrac{1}{8}x\left(63x^4 - 70x^2 + 15\right). \tag{27}$$

Written in terms $x = \cos\theta$ (commonly written $\mu = \cos\theta$), the first few become

$$P_0^0(\cos\theta) = 1 \tag{28}$$

$$P_1^0(\cos\theta) = \cos\theta \tag{29}$$

$$P_1^1(\cos\theta) = -\sin\theta \tag{30}$$

$$P_2^0(\cos\theta) = \tfrac{1}{2}\left(3\cos^2\theta - 1\right) \tag{31}$$

$$P_2^1(\cos\theta) = -3\sin\theta\cos\theta \tag{32}$$

$$P_2^2(\cos\theta) = 3\sin^2\theta \tag{33}$$

$$P_3^0(\cos\theta) = \tfrac{1}{2}\cos\theta\left(5\cos^2\theta - 3\right) \tag{34}$$

$$P_3^1(\cos\theta) = -\tfrac{3}{2}\left(5\cos^2\theta - 1\right)\sin\theta \tag{35}$$

$$P_3^2(\cos\theta) = 15\cos\theta\sin^2\theta \tag{36}$$

$$P_3^3(\cos\theta) = -15\sin^3\theta. \tag{37}$$

The derivative about the origin is

$$\left[\frac{d\,P_\nu^\mu(x)}{d\,x}\right]_{x=0} = \frac{2^{\mu+1}\sin\left[\frac{1}{2}\pi(\nu+\mu)\right]\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu+1\right)}{\pi^{1/2}\,\Gamma\left(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2}\right)} \tag{38}$$

and the logarithmic derivative is

$$\left[\frac{d\ln P_\lambda^\mu(z)}{d\,z}\right]_{z=0} = 2\tan\left[\tfrac{1}{2}\pi(\lambda+\mu)\right]\frac{\left[\frac{1}{2}(\lambda+\mu)\right]!\left[\frac{1}{2}(\lambda-\mu)\right]!}{\left[\frac{1}{2}(\lambda+\mu-1)\right]!\left[\frac{1}{2}(\lambda-\mu-1)\right]!}. \tag{39}$$

4.6 SUMMARY

- We study The Bessel differential equation is the linear second-order ordinary differential equation given by

  $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right)y = 0.$$

- We learn The Legendre differential equation is the second-order ordinary differential equation

  $$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0,$$

- We study Bessel Neumann series of the form

  $$\sum_{n=0}^{\infty} a_n J_{\nu+n}(z),$$

- We study the Bessel polynomials $\{y_n(x, a, b)\}_{n=0}^{\infty}$ satisfy

  $$x^2y'' + (ax+b)y_n' - n(n+a-1)y = 0$$

- We study the The associated Legendre polynomials $P_l^m(x)$ and $P_l^{-m}(x)$ generalize the Legendre polynomials $P_l(x)$ and are solutions to the associated Legendre differential equation.

4.7 KEYWORD

Neumann Series : A Neumann series is a mathematical series of the form. where T is an operator. Here, T^k is a mathematical notation for k consecutive operations of the operator T. This generalizes the geometric series

Bessel Series : Bessel's differential equation that are finite at the origin (x = 0) for integer or positive α and diverge as x approaches zero for negative non-integer α.

Lengdre's Polynomial : The Legendre polynomials, sometimes called Legendre functions of the first kind, Legendre coefficients, or zonal harmonics, are solutions to the Legendre differential equation. If is an integer, they are polynomials

4.8 EXERCISE

Q. 1 Define Legendre's Polynomial .

Q. 2 Define Bessel First Order Differential Equation.

Q. 3 Find the general solution to

$$x^2y'' - xy' + y = 0 .$$

Q. 4 Define Legendre's differential equation of First Kind.

Q. 5 Define Legendre's differential equation of Second Kind.

4.9 ANSWER TO CHECK IN PROGRESS

Check In Progress-1

Answer Q. 1 Check in section 3

Q. 2 Check in section 3.1

Check In progress-II

Answer Q. 1 Check in section 6

Q. 2 Check in section 6.1

4.10 SUGGESTION READING AND REFERENCES

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UNIT 5 TOPIC : EXISTENCE AND UNIQUENESS OF INITIAL VALUE PROBLEMS

STRUCTURE

5.0 OBJECTIVE

- We study in this unit existence and uniqueness of solutions
- We study Picard iterative process
- We study taylor polynomials and its examples

- We study picard's existence theorem and its proof
- We study second degree taylor polynomials with its examples
- We study exact and non-exact equations

5.1 INTRODUCTION

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order, so fractional differential equations have wider application. Fractional differential equations have gained considerable importance; it can describe many phenomena in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetic.

In the recent years, there has been a significant development in fractional calculus and fractional differential equations; see Kilbas et al. , Miller and Ross , Podlubny , Baleanu et al. , and so forth. Research on the solutions of fractional differential equations is very extensive, such as numerical solutions, see El-Mesiry et al. and Hashim et al. , mild solutions, see Chang et al. and Chen et al. , the existence and uniqueness of solutions for initial and boundary value problem, and so on.

With the deep study, many papers that studied the fractional equations contained more than one fractional differential operator.

5.2 EXISTENCE AND UNIQUENESS OF SOLUTIONS

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a first order differential equation which satisfies a given initial condition. How does it work? Why is it the case? We believe it but it would be interesting to see the main ideas behind. First let us state the theorem itself.

Theorem. Let $f(x,y)$ be a real valued function which is continuous on the rectangle $R = \{(x,y); |x - x_0| \leq a\ ,\ |y - y_0| \leq b\}$.

Assume f has a partial derivative with respect to y and that $\frac{\partial f}{\partial y}$ is also continuous on the rectangle *R*. Then there exists an interval $I = [x_0 - h, x_0 + h]$ (with $h \leq a$) such that the initial value problem

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0 \end{cases}$$

has a unique solution *y*(*x*) defined on the interval *I*.

Note that the number *h* may be smaller than *a*. In order to understand the main ideas behind this theorem, assume the conclusion is true. Then if *y*(*x*) is a solution to the initial value problem, we must have

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$$

It is not hard to see in fact that if a function *y*(*x*) satisfies the equation (called functional equation)

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt$$

on an interval *I*, then it is solution to the initial value problem

$$y' = f(x, y) \quad y(x_0) = y_0$$

Picard was among the first to look at the associated functional equation. The method he developed to find *y* is known as the method of successive approximations or Picard's iteration method. This is how it goes:

Step 1. Consider the constant function

$$y_0(x) = y_0$$

Step 2. Once the function $y_n(x)$ is known, define the function

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt$$

Step 3. By induction, we generate a sequence of functions $\{y_n(x)\}$ which, under the assumptions made on $f(x,y)$, converges to the solution $y(x)$ of the initial value problem

$$y' = f(x,y) \quad y(x_0) = y_0$$

Example: Suppose the differential equation $\frac{dy}{dt} = f(t,y)$ satisfies the Existence and Uniqueness Theorem for all values of y and t.

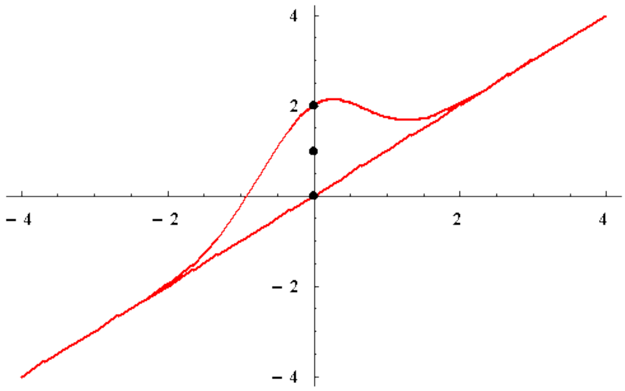
Suppose $y_1(t) = t + 2e^{-t^2}$ and $y_2(t) = t$ are two solutions to this differential equation.

1.What can you say about the behavior of the solution of the solution $y(t)$ satisfying the initial condition $y(0)=1$?

Hint: Draw the two solutions y_1 and y_2.

2.Address the behavior of $y(t)$ as t approaches $+\infty$, and as t approaches $-\infty$.

Solution: 1. First let us draw the graphs of y_1 and y_2.



Since we have $0 = y_2(0) < y(0) < y_1(0) = 2$, we deduce from the Existence and Uniqueness Theorem that for all t, we have

$$y_2(t) < y(t) < y_1(t) .$$

In particular, $y(t)$ has the line $y=t$ as an oblique asymptote which answers the second question.

We cannot predict that $y(t)$ is an increasing function.

5.3 PICARD ITERATIVE PROCESS

Indeed, often it is very hard to solve differential equations, but we do have a numerical process that can approximate the solution. This process is known as the **Picard iterative process**.

First, consider the IVP

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \ .$$

It is not hard to see that the solution to this problem is also given as a solution to (called the integral associated equation)

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s)ds \ .$$

The Picard iterative process consists of constructing a sequence (y_n) of functions which will get closer and closer to the desired solution. This is how the process works:

(1) $y_0(x) = y_0$ for every x;

(2)then the recurrent formula holds

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt,$$

for $n \geq 1$.

Example: Find the approximated sequence (y_n), for the IVP

$$y' = 2x(1 + y) \quad y(0) = 0.$$

Solution: First let us write the associated integral equation

Notes

$$y(x) = \int_0^x 2s(1 + y(s))ds \,.$$

Set $y_0(x) = 0$. Then for any $n \geq 1$, we have the recurrent formula

$$y_{n+1}(x) = \int_0^x 2s(1 + y_n(s))ds \,.$$

We have $y_1(x) = \int_0^x 2s\,ds = x^2$, and

$$y_2 = \int_0^x 2s(1 + s^2)ds = x^2 + \frac{x^4}{2} \,.$$

We leave it to the reader to show that

$$y_n(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + ... + \frac{x^{2n}}{n!} \,.$$

We recognize the Taylor polynomials of (which also get closer and closer to) the function

$$y(x) = e^{x^2} - 1 \,.$$

Taylor Polynomials

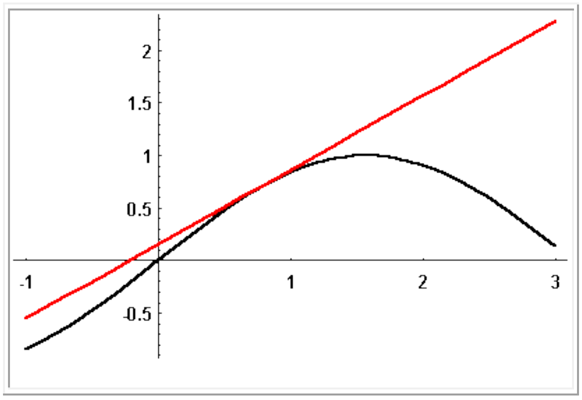
Introduction

The fundamental idea in differential calculus is that a function can be ``locally" approximated by its tangent line.

For instance consider the function $\sin(x)$ near $x_0 = \pi/4$. Since its derivative at $\pi/4$ equals $\sin'(\pi/4) = \cos(\pi/4)$, the tangent line at $x_0 = \pi/4$ can be written as

$$y(x) = \sin(\pi/4) + \cos(\pi/4) \cdot (x - \pi/4).$$

In the picture below, the sine function is black, while its tangent line is depicted in red. Close to $\pi/4$, both are quite close!



Example : Find an equation for the tangent line of the function $f(t) = \frac{1}{t}$ **at the point** $t_0 = 2$.

Answer : The derivative of *f*(*t*) is $f'(t) = -\frac{1}{t^2}$, thus $f'(2) = -\frac{1}{4}$.

Since $f(t) = \frac{1}{2}$, we obtain as an equation for the tangent line at $t_0 = 2$:

$$y(t) = \frac{1}{2} - \frac{1}{4}(t - 2).$$

No reason to only compute second degree Taylor polynomials! If we want to find for example the fourth degree Taylor polynomial for a function *f*(*x*) with a given center x_0, we will insist that the polynomial and *f*(*x*) have the same value and the same first four derivatives at x_0.

A calculation similar to the previous one will yield the formula:

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{2 \cdot 3}(x$$

Some more notation.

Notes

(1) Usually Taylor polynomials are denoted by $T_n(x)$, where *n* indicates its degree.

(2) The factorial sign comes in handy: Recall that $n! := 1 \cdot 2 \cdot 3 \cdots n$. For instance $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$ Mathematicians usually say that 0!=1.

(3) Recall that one writes $f^{(n)}(x)$ for the *n*th derivative of the function*f*. $f^{(0)}(x)$ just means *f*(*x*).

With these notations, we can then write the *n*th term of a Taylor polynomial as

$$\frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Thus we obtain the general formula for the *n*th Taylor polynomial of a function *f*(*x*) with center x_0:

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots$$

An alternative way of writing this--for the not easily *Mathematese*-intimidated-- is provided by the summation notation:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

High time to try it yourself!

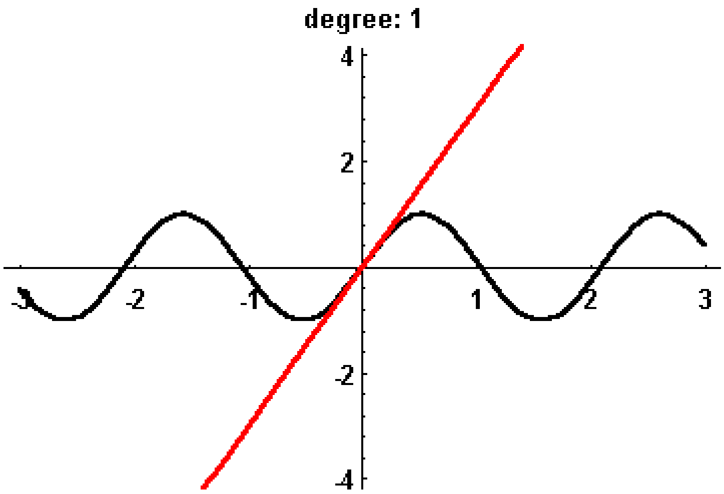
Example : Find the 5th degree Taylor polynomial for the function $f(x) = \sin(3x)$ with center $x_0 = 0$.

Answer : $$T_5(x) = 3x - \frac{27}{6}x^3 + \frac{243}{120}x^5.$$

As you can see in the following animated picture for the example of this sine function, increasing the degree of the Taylor polynomial for a given function *f*(*x*) at a given point x_0 generally has two effects:

- At a given point *x*, the approximation of *f*(*x*) by the value of the Taylor polynomial $T_n(x)$ becomes more accurate.
- The approximation becomes ``good'' over a larger interval around the center x_0. (This is really saying the same as the previous statement!)

The function *sin*(3*x*) is black, while its Taylor polynomials with center $x_0 = 0$ are shown in red.



Example : Find the Taylor polynomial for the following functions and centers:

1. $f(x) = \ln x$, center $x_0 = 1$, degree 5.
2. $f(x) = \tan x$, center $x_0 = 0$, degree 3.
3. $f(x) = \dfrac{1}{x^2 + 1}$, center $x_0 = 0$, degree 3.

Answer : If you did the last problem, you realize that you only have to compute the first 2 derivatives, since the function under consideration is even, and thus its Taylor polynomial has only even powers.

Notes

$$
\begin{aligned}
f(x) &= \frac{1}{x^2+1} \\
f'(x) &= \frac{-2x}{(x^2+1)^2} \\
f''(x) &= \frac{-2(x^2+1)^2 + 8x^2(x^2+}{(x^2+1)^4},
\end{aligned}
$$

resulting in *f*(0)=1, *f*'(0)=0 and *f*''(0)=-2. Consequently the Taylor polynomial looks like this:

$$T_3(x) = 1 - x^2.$$

Check In Progress-I

Q. 1 Define Picard Iterative Process.

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

Q. 2 Find the approximated sequence (y_n), for the IVP

$$y' = 2x(1+y) \quad y(0) = 0.$$

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

5.4 PICARD'S EXISTENCE THEOREM

If f is a continuous function that satisfies the Lipschitz condition

$$|f(x,t) - f(y,t)| \le L|x-y| \tag{1}$$

in a surrounding of $(x_0, t_0) \in \Omega \subset \mathbb{R}^n \times \mathbb{R} = \{(x, t) : |x - x_0| < b, |t - t_0| < a\}$, then the differential equation

$$\frac{dx}{dt} = f(x, t) \tag{2}$$

$$x(t_0) = x_0 \tag{3}$$

has a unique solution $x(t)$ in the interval $|t - t_0| < d$, where $d = \min(a, b/B)$, min denotes the minimum, $B = \sup|f(t, x)|$, and sup denotes the supremum.

5.4.1 Peano Derivative

One of the generalizations of the concept of a derivative. Let there exist a $\delta > 0$ such that for all t with $|t| < \delta$ one has

$$f(x_0 + t) = \alpha_0 + \alpha_1 t + \ldots + \frac{\alpha_r}{r!} t^r + \gamma(t) t^r,$$

where $\alpha_0, \ldots, \alpha_r$ are constants and $\gamma(t) \to 0$ as $t \to 0$; let $\gamma(0) = 0$. Then α_r is called the generalized Peano derivative of order r of the function f at the point x_0. Symbol: $f_{(r)}(x_0) = \alpha_r$; in particular, $\alpha_0 = f(x_0)$, $\alpha_1 = f_{(1)}(x_0)$. If $f_{(r)}(x_0)$ exists, then $f_{(r-1)}(x_0)$, $r \geq 1$, also exists. If the finite ordinary two-sided derivative $f^{(r)}(x_0)$ exists, then $f_{(r)}(x_0) = f^{(r)}(x_0)$. The converse is false for $r > 1$: For the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \text{ and rational,} \\ 0, & x = 0 \text{ or irrational,} \end{cases}$$

one has $f_{(r)}(0) = 0$, $r = 1, 2, \ldots$, but $f_{(1)}(x)$ does not exist for $x \neq 0$ (since $f(x)$ is discontinuous for $x \neq 0$). Consequently, the ordinary derivative $f^{(r)}(0)$ does not exist for $r > 1$.

Infinite generalized Peano derivatives have also been introduced. Let for all t with $|t| < \delta$,

$$f(x_0 + t) = \alpha_0 + \alpha_1 t + \ldots + \frac{\alpha_r(t)}{r!} t^r,$$

where α_0 ,..., α_{r-1} are constants and $\alpha_r(t) \to \alpha_r$ as $t \to 0$ (α_r is a number or the symbol ∞). Then α_r is also called the Peano derivative of order r of the function f at the point x_0. It was introduced by G. Peano.

Theorem 1.1 (Cauchy-Peano Existence Theorem)

Let $f:\mathbb{R}^2 \to \mathbb{R}$ *b continuous in a neighborhood of the point* $(x_0, y_0) \in \mathbb{R}^2$*. Then there exists a* $\alpha > 0$ *such that the IVP*

$$y' = f(x, y),\ y(x_0) = y_0 \tag{1}$$

has a solution ϕ *on the interval* $I := [x_0 - \alpha, x_0 + \alpha]$*. That is, there exists a* $\phi = \phi(x)$ *defined on* I *such that*

$$\phi'(x) = f(x, \phi(x)),\ x \in I$$

and $\phi(x_0) = y_0$.

Remark: In general, ϕ is not unique. If f is Lipschitz continuous with respect to y, then uniqueness follows from the Picard theorem / Picard iterates.

Remark 2: We give two proofs to show the differences in the two approaches. The first one is the approximation procedure, and the second is the topological / fixed point method.

Proof 1: Since f is continuous in a neighborhood of (x_0, y_0), there exists $a > 0$ such that f is continuous in the closed square

$$Q := \{(x, y) \in \mathbb{R}^2 : |x - x_0| \le a, |y - y_0| \le a\}.$$

Let $M = \max_Q |f(x, y)|$ (which exists as f is a continuous function on a compact set). Set $\alpha := \frac{a}{M}$ (we have assumed, WLOG that $M \ge 1$).

We know, from previous courses, that $\phi = \phi(x)$ is a solution of the IVP (1) if and only if ϕ satisfies the integral equation

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))dt \qquad (2)$$

(Consequence of FTC)

Since f is uniformly continuous on Q (a compact set), given $\epsilon > 0$,

there exists a $\delta = \delta(\epsilon)$ such

that $|x - \hat{x}| < \delta$ and $|y - \hat{y}| < \delta$ implies $|f(x,y) - f(\hat{x},\hat{y})| < \epsilon$ for

all $(x,y), (\hat{x},\hat{y}) \in Q$. Therefore, let $\epsilon = \epsilon_n = \frac{1}{n}$ and $\delta = \delta_n$.

We choose

points $x_j^{(n)}$ with $x_0^{(n)} = x_0$ and $x_{k(n)} = x_0 + \alpha$ (where $0 \le j \le k(n)$

), with $|x_{j+1}^{(n)} - x_j^{(n)}| \le \frac{\delta_n}{M}$.

We define the polygonal

approximation ϕ_n on $[x_0, x_0 + \alpha]$ by $\phi_n(x_0) = y_0$ and

$\phi_n'(x) = f(x_0, y_0), x_0 \le x \le x_1^{(n)}$. Then, $y_1^{(n)} = \phi_n(x_1^{(n)})$, and we

continue this iterative definition to get

that $\phi_n'(x) = f(x_j^{(n)}, y_j^{(n)})$ for $x_j^{(n)} \le x \le x_{j+1}^{(n)}$.

Note that ϕ_n is piecewise C^1 (continuous and has, perhaps, a jump discontinuity in the derivative at the partition points). We then define

$$\Delta_n(t) = \begin{cases} \phi_n'(t) - f(t, \phi_n(t)), & x_j^{(n)} < t < x_{j+1}^{(n)} \\ 0, & t = x_j^{(n)} \end{cases}$$

Then notice that

$$\phi_n(x) = y_0 + \int_{x_0}^{x} \phi_n'(t)dt = y_0 + \int_{x_0}^{x} [f(t, \phi_n(t)) + \Delta_n(t)]\, dt$$

We claim that $|\Delta_n(t)| < \frac{1}{n}$. Note,

$$\phi_n'(t) = f(x_j^{(n)}, y_j^{(n)}), t \in [x_j^{(n)}, x_{j+1}^{(n)}]$$

Therefore,

$$\Delta_n(t)| = |f(x_j^{(n)}, y_j^{(n)}) - f(t, \phi_n(t))|, t \in [x_j^{(n)}, x_{j+1}^{(n)}]$$

Further note

$$|t - x_j^{(n)}| \leq |x_{j+1}^{(n)} - x_j^{(n)} \leq \delta_n$$

$$|y_j^{(n)} - \phi_n(t)| \leq |y_j^{(n)} - y_{j+1}^{(n)}| \leq M|x_j^{(n)} - x_{j+1}^{(n)}|$$

$$\leq \delta_n$$

Thus, from our $\epsilon - \delta$ condition, $|\Delta_n(t)| \leq \epsilon_n = \frac{1}{n}$.

By the Ascoli-Arzela Theorem, there exists a uniformly convergent subsequence of $\{\phi_n\}$ which converges to $\phi(t) = \lim_{k\to\infty} \phi_{n_k}(t)$ for $t \in [x_0 - \alpha, x_0 + \alpha]$. That is,

$$\phi_{n_k}(x) = y_0 + \int_{x_0}^{x} [f(t, \phi_{n_k}(t)) + \Delta_{n_k}(t)]\, dt$$

$$\phi(t) = y_0 + \int_{x_0}^{x} f(t, \phi(t)) dt,$$

which is exactly what we needed to show a solution to the DE ☐.

Proof 1, Variant 2:

Recall, from the first proof,

$$Q := \{(x, y) : |x - x_0| \leq a, |y - y_a| \leq a\} \subset U$$

where U is a neighborhood of (x_0, y_0), throughout which f is continuous.

$$M := \max_Q |f(x, y)|$$

$$\alpha := \frac{a}{M}$$

We define the sequence Ψ_n as follows:

$$\Psi_n(x) := \begin{cases} y_0, & x \le x_0 \\ y_0 + \int_{x_0}^{x} f(t, \Psi_n(t - \frac{\alpha}{n})), & x_0 \le x \le x_0 + \alpha \end{cases}$$

(where $n \ge 1$). This is well defined - on the interval $[x_0, x_0 + \frac{\alpha}{n}]$, $x - \frac{\alpha}{n} \le x_0$, so Ψ_n is defined

So again, we have a sequence $\{\Psi_n(x)\}$ defined on $[x_0, x_0 + \alpha]$. By the Ascoli-Arzela theorem, there exists a uniformly convergent subsequence - say it converges to $\Psi(x)$.

$$\Psi_{n_k}(x) = y_0 + \int_{x_0}^{x} f(t, \Psi_{n_k}(t - \frac{\alpha}{n_k}))dt$$

$$\Psi(x) = y_0 + \int_{x_0}^{x} f(t, \Psi(t))dt$$

It follows that Ψ is also a solution to the DE.

Remarks on Uniqueness

Note again that the system (1) is equivalent to the integral equation (2). If we have a Lipschitz condition, then we can use the Picard iterates method on the integral equation to get a unique solution. We define

$$y_0(x) = y_0$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t))dt.$$

As we commented above, this converges to a unique solution of 1 if f is Lipschitz in y.

5.5SECOND DEGREE TAYLOR POLYNOMIALS

One way to see that the tangent line to a function *f*(*x*) at a given point x_0 is the best line approximating the function is to observe that the tangent line is the (only) line passing through the point $(x_0, f(x_0))$ and having the same slope as $f(x)$ at x_0.

So what about about finding the "best" parabola approximating the function $f(x)$ near x_0? We should look for the parabola passing through $(x_0, f(x_0))$, which has the same slope (the first derivative) as $f(x)$ at x_0, **and** which has the same second derivative as $f(x)$ at x_0!

Let's try it: Consider $f(x) = e^{-(x-1)}$ near $x_0 = 1$. The parabola we are trying to find has the generic form:

$$p(x) = a + b(x - 1) + c(x - 1)^2.$$

Writing the parabola this way, it is easier to compute its derivatives at $x_0 = 1$: $p'(x)=b+2c(x-1)$ and $p''(x)=2c$. Substituting $x_0 = 1$ we obtain:

$$p(1) = a \ , \ p'(1) = b \ , \ p''(1) = 2c.$$

Recall, we want to find the parabola which has the same derivatives at x_0 as $f(x)$. This yields the conditions:

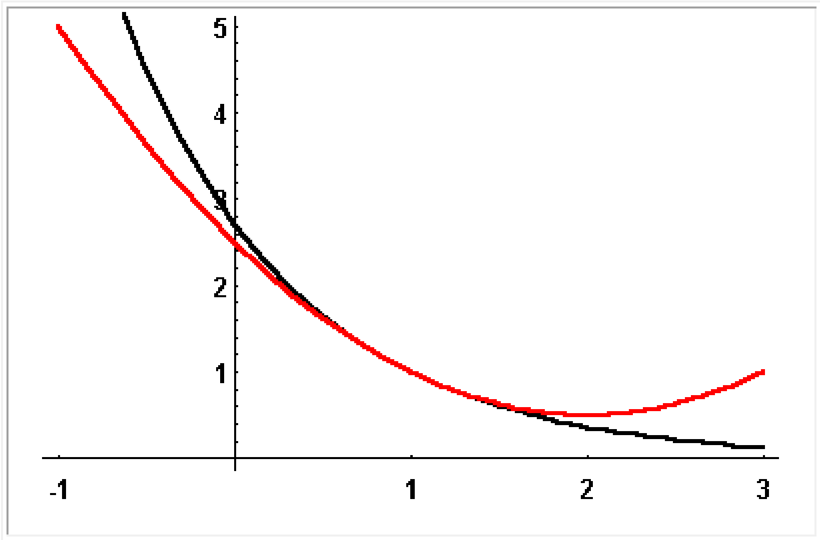
$$\begin{aligned} f(1) &= \\ f'(1) &= \\ f''(1) &= \end{aligned}$$

Now $f(1) = e^{-0} = 1$; $f'(1) = -e^{-0} = -1$ and $f''(1) = e^{-0} = 1$. Solving for the coefficients and substituting in the formula for $p(x)$, we obtain

$$p(x) = 1 - (x - 1) + \frac{1}{2}(x - 1)^2.$$

The polynomial $p(x)$ is called the **second degree Taylor polynomial** of the function $f(x) = e^{-(x-1)}$ at the point $x_0 = 1$.

The picture below shows *f*(*x*) in black and its second degree Taylor polynomial at $x_0 = 1$ in red.



It is not hard to see what the general formula will look like: If we replace $x_0 = 1$ by a "general" x_0 above, we obtain:

$$p(x) = a + b(x - x_0) + c(x - x_0)^2$$

as the general form of the Taylor polynomial at x_0; We need that

$$\begin{aligned} f(x_0) &= \\ f'(x_0) &= \\ f''(x_0) &= \end{aligned}$$

and consequently

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2.$$

x_0 is called the **center** of the Taylor polynomial. Note: The center x_0 is fixed, the variable name for the polynomial is *x*. Even if we consider the same function *f*(*x*), different centers will usually yield different Taylor polynomials (just as a function usually has different tangent lines at various points!).

Try it yourself!

Example : Find the quadratic Taylor polynomial for the function $f(x) = \frac{1}{x}$ with the center $x_0 = -2$.

Answer : The relevant information is listed in the following table

$$f(x) = \frac{1}{x} \qquad f(-2) = -$$
$$f'(x) = -\frac{1}{x^2} \qquad f'(-2) = -$$
$$f''(x) = \frac{2}{x^3} \qquad f''(-2) =$$

Using the formula above we obtain as the second degree Taylor polynomial:

$$p(x) = -\frac{1}{2} - \frac{1}{4}(x+2) - \frac{1}{8}(x+2)^2.$$

5.6 NUMERICAL TECHNIQUE: EULER'S METHOD

The same idea used for slope fields--the graphical approach to finding solutions to first order differential equations--can also be used to obtain numerical approximations to a solution. The basic idea of differential calculus is that, close to a point, a function and its tangent line do not differ very much. Consider, for example, the function $f(x) = \sin(x)$, and its tangent line at $x = \pi/6$.

Now consider the differential equation

$$y' = x - y.$$

If we want to compute the solution passing through the point (-1,4), then we can compute the tangent line at this point. Its slope at *x*=-1 is given by the differential equation

$$y'(0) = -1 - 4 = -5;$$

thus, the equation for the tangent line is given by

$$y(x) = 4 - 5(x+1).$$

Since we expect the solution to the differential equation and its tangent line to be close when *x* is close to -1, we should also expect that the solution to the differential equation at, let's say, *x*=-0.75 will be close to the tangent line at *x*=-0.75.

We compute the *y*-value of the tangent line to be *y*(-0.75)=2.75. This method can now be iterated; the tangent line equation for *x*=-.075 and *y*=2.75 is given by

$$y(x) = 2.75 - 3.5(x+0.75);$$

using the tangent line equation for *x*=-0.5, we obtain as an approximation to our solution

$$y(-0.5) = 1.875.$$

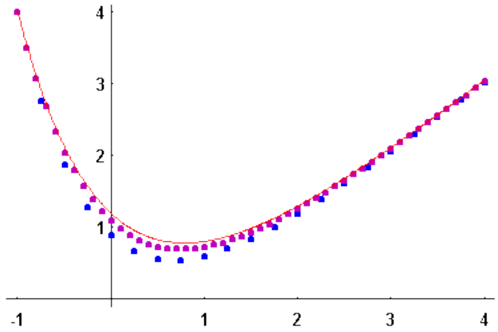
What is the formula we use to find our approximations?

We start at the point (x_0, y_0). Let *h* denote the *x*-increment. Then $x_1 = x_0 + h$. y_1 is the the *y*-coordinate of the point on the line passing through the point (x_0, y_0) with slope $y'(x_0) = f(x_0, y_0)$; thus $y_1 = y_0 + h \cdot f(x_0, y_0)$.

The next approximation is found by replacing x_0 and y_0 by x_1 and y_1; so $x_2 = x_1 + h$ and $y_2 = y_1 + h \cdot f(x_1, y_1)$. In general, we obtain the following formula for $n = 1, 2, \ldots$,

$$x_n = x_{n-1} + h = x_0 + n \cdot$$
$$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_n$$

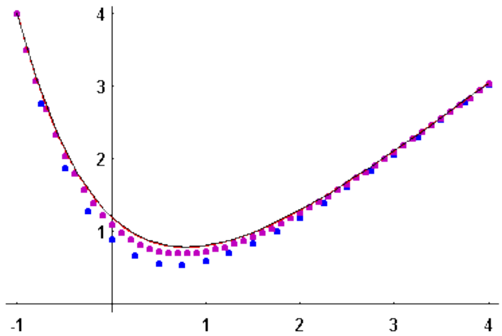
We obtain better approximations if we reduce the step size *h*. The following graphs give approximations for step sizes *h*=0.25 in blue, *h*=0.1 in purple and *h*=0.01 in red:



For this example it is not hard to compute the exact solution

$$y(x) = -1 + 6 \cdot e^{-(1+x)} + x.$$

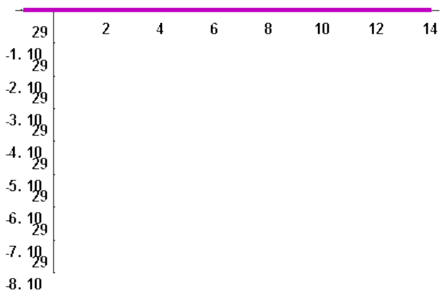
The next graph shows the exact solution in black. We see that a step size of *h*=0.01, leading to 500 steps of computation, does a satisfactory job in approximating the exact solution.



The rest of this page is devoted to some of the pitfalls of numerical computations. Here is another example of a "harmless" differential equation:

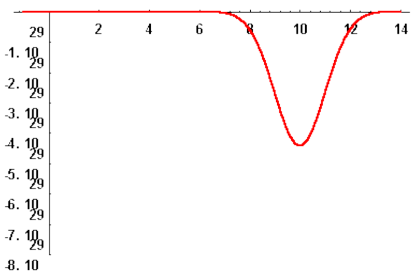
$$y' = -(x - 10) \cdot y.$$

First, we try Euler's method with a step size of *h*=0.1. Recall that this step size gave a satisfactory approximation in the previous example.

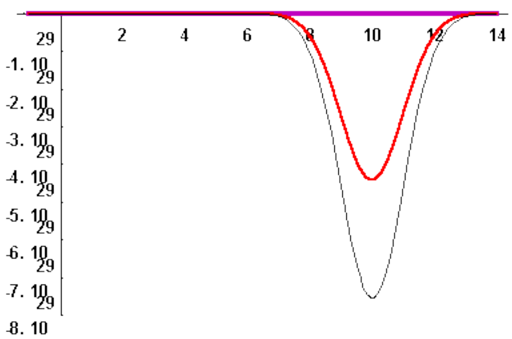


Not much is happening. Note, however, that the scale on the *y*-axis is of the magnitude 10^{29}.

Let's try again with a step size of *h*=0.0025, leading to 6,000 computing steps.



It is easy to solve the differential equation analytically (do it!). The next picture compares the previous approximation (in red) to the graph of the exact solution (in black). The approximation is off by about 50%!



It is of interest to use numerical methods only when one is unable to compute solutions with pencil and paper. But, in such a situation you cannot compare the approximation to the exact solution so you have no control over how good your approximation is! If you take a course in Numerical Mathematics you will learn that there are ways to predict the error in Euler's method even if you cannot compute the exact solution.

Notes

The class is also the place to learn more sophisticated numerical methods to solve differential equations. For the differential equation we considered, one should use a method which automatically reduces the step size where the slope of the solution changes rapidly!

The last example addresses another pitfall. Let's consider the differential equation

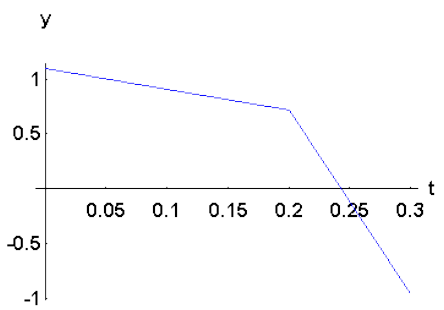
$$y' = 50(y-1)^2(y-5).$$

We want to find the solution satisfying the initial condition $y(0)=1.1$, using Euler's method with step size $h=.1$.

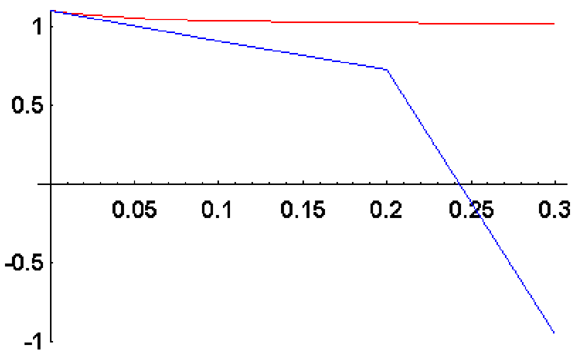
The following is a table with the first four values; the function is decreasing rapidly:

t	$y(t)$
0	1.1
0.1	0.905
0.2	0.720213
0.3	−0.95491
0.4	−114.744

Let's look at the graph of the approximation.



The next picture compares the approximation (in blue) to the graph of the exact solution (in red).



Notice that there is no hint of a resemblance between the two. What has happened? An explanation requires the knowledge of the concept of a node. The differential equation has an equilibrium solution *y*=1. The equilibrium is a down node; thus the exact solution with initial condition *y*(0)=1.1 is decreasing and approaches 1 as *t* approaches infinity. The Euler approximation "jumps" below the equilibrium solution in the first computational step (look again at the table above). Once it is below the equilibrium solution it moves away from the position of the node rather rapidly! In our case, a smaller step size would prevent the Euler approximation from ever jumping below the equilibrium solution.

Example: Consider the autonomous differential equation with the initial condition

$$\frac{dy}{dt} = 18y^2, \quad y(0) = -0.1.$$

1.Find

$$\lim_{t \to +\infty} y(t).$$

Hint: You may want to sketch the Slope Fields of this differential equation

2.Find the first five terms of the Euler Approximation when $\Delta t = 1$.

3.Is there a contradiction between the results of 1 and 2 ? If yes, explain what happened.

Notes

Solution : 1. Since $\frac{dy}{dt} \geq 0$, any solution to the differential equation is increasing. This differential equation has one critical solution *y*=0. Since the initial condition satisfied by the solution to the IVP is *y*(0) = -0.1 < 0, then we have *y*(*t*) <0 for all *t*. We deduce from this that

$$\lim_{t \to +\infty} y(t) = 0 \ .$$

2.Recall the formula which gives the euler's approximations to the solution

$$y_0 = y(0) = -0.1 \ \text{ and } \ y_{n+1} = y_n + \Delta t \times 18 y_n^2 = 3$$

This gives the first five terms as:

$$\begin{array}{lll} y_0 = -0.1 & y_1 = 0.08 & y_2 = 0.1952 \\ y_3 = 0.881055 & y_4 = 14.8537 & y_5 = 3986.23 \end{array}$$

3.According to the result in Part 1, the solution to the given IVP should always be negative and according to the above Euler's approximation the first term y_1 is positive. This is our contradiction. As a matter of fact, according to the slope field, the Euler's approximation should continue to rise and even tend to $+\infty$, which even further contradicts the conclusion of 1.

4 The reason behind this is that $\Delta t = 1$ is a large step. So, after the first shot, it shoots above the critical solution *y*=0. Try to do the same problem with different step-sizes.

Check In Progress-II

Q. 1 State Picard Existence Theorem.

Solution : . . . .

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. . . .

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. . . .

Q. 2 Define Euler's Method.

Solution : . . . .

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5.7 EXACT AND NONEXACT EQUATIONS

All the techniques we have reviewed so far were not of a general nature since in each case the equations themselves were of a special form. So, we may ask, what to do for the general equation

$$\frac{dy}{dx} = f(x,y)\;.$$

Let us first rewrite the equation into

$$(E) \quad M(x,y)dx + N(x,y)dy = 0.$$

This equation will be called **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.,$$

and **nonexact** otherwise. The condition of exactness insures the existence of a function $F(x,y)$ such that

$$\begin{cases} \dfrac{\partial F}{\partial x} = M(x,y), \\ \dfrac{\partial F}{\partial y} = N(x,y). \end{cases}$$

When the equation (E) is exact, we solve it using the following steps:

(1) Check that the equation is indeed exact;

(2) Write down the system

$$\begin{cases} \dfrac{\partial F}{\partial x} = M(x,y), \\ \dfrac{\partial F}{\partial y} = N(x,y). \end{cases}$$

(3) Integrate either the first equation with respect of the variable *x* or the second with respect of the variable *y*. The choice of the equation to be integrated will depend on how easy the calculations are. Let us assume that the first equation was chosen, then we get

$$F(x,y) = \int M(x,y)dx + \theta(y).$$

The function $\theta(y)$ should be there, since in our integration, we assumed that the variable *y* is constant.

(4) Use the second equation of the system to find the derivative of $\theta(y)$. Indeed, we have

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \theta'(y) = N(x,y)$$

,

which implies

$$\theta'(y) = N(x,y) - \frac{\partial}{\partial y}\left(\int M(x,y)dx\right).$$

Note that θ is a function of *y* only. Therefore, in the expression giving $\theta'(y)$ the variable, *x*, should disappear. Otherwise something went wrong!

(5) Integrate to find $\theta(y)$;
(6) Write down the function *F*(*x*,*y*);
(7) All the solutions are given by the implicit equation

$$F(x,y) = C.$$

(8) If you are given an IVP, plug in the initial condition to find the constant *C*.

You may ask, what do we do if the equation is not exact? In this case, one can try to find an integrating factor which makes the given differential equation exact.

5.8 INTEGRATING FACTOR TECHNIQUE

Assume that the equation

$$M(x,y)dx + N(x,y)dy = 0,$$

is not exact, that is-

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

In this case we look for a function *u*(*x,y*) which makes the new equation

$$(E^*) \qquad u(x,y)M(x,y)dx + u(x,y)N(x,y)dy = 0,$$

an exact one. The function *u*(*x,y*) (if it exists) is called the **integrating factor**. Note that *u*(*x,y*) satisfies the following equation:

$$\frac{\partial M}{\partial y}u + \frac{\partial u}{\partial y}M = \frac{\partial N}{\partial x}u + \frac{\partial u}{\partial x}N \ .$$

This is not an ordinary differential equation since it involves more than one variable. This is what's called a partial differential equation. These types of equations are very difficult to solve, which explains why the determination of the integrating factor is extremely difficult except for the following two special cases:

Case 1: There exists an integrating factor *u*(*x*) function of *x* only. This happens if the expression

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N},$$

is a function of *x* only, that is, the variable *y* disappears from the expression. In this case, the function *u* is given by

$$u(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx\right).$$

Case 2: There exists an integrating factor *u*(*y*) function of *y* only. This happens if the expression

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M},$$

is a function of *y* only, that is, the variable *x* disappears from the expression. In this case, the function *u* is given by

$$u(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} dy\right).$$

Once the integrating factor is found, multiply the old equation by *u* to get a new one which is exact. Then you are left to use the previous technique to solve the new equation.

Advice: if you are not pressured by time, check that the new equation is in fact exact!

Let us summarize the above technique. Consider the equation

$$M(x, y)dx + N(x, y)dy = 0.$$

If your equation is not given in this form you should rewrite it first.

Step 1: Check for exactness, that is, compute

$$\frac{\partial M}{\partial y} \text{ and } \frac{\partial N}{\partial x},$$

then compare them.

Step 2: Assume that the equation is not exact (if it is exact go to step ?). Then evaluate

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}.$$

If this expression is a function of *x* only, then go to step 3. Otherwise, evaluate

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}.$$

If this expression is a function of *y* only, then go to step 3. Otherwise, you can not solve the equation using the technique developped above!

Step 3: Find the integrating factor. We have two cases:

3.1 If the expression $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of *x* only. Then an integrating factor is given by

$$u(x) = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx\right).$$
;

3.2 If the expression $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of *y* only, then an integrating factor is given by

$$u(y) = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} dy\right).$$

Step 4: Multiply the old equation by *u*, and, if you can, check that you have a new equation which is exact.

Step 5: Solve the new equation using the steps described in the previous section.

The following example illustrates the use of the integrating factor technique:

Integrating Factor Technique:

Example : Find all the solutions to

$$\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}.$$

Solution: Note that this equation is in fact homogeneous. But let us use the technique of exact and nonexact to solve it. Let us follow these steps:

(1) We rewrite the equation to get

$$(3xy + y^2)dx + (x^2 + xy)dy = 0.$$

Hence, $M(x,y) = 3xy + y^2$ and $N(x,y) = x^2 + xy$.

(2) We have

$$\frac{\partial M}{\partial y} = 3x + 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x + y,$$

which clearly implies that the equation is not exact.

(3) Let us find an integrating factor. We have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1}{x}.$$

Therefore, an integrating factor *u*(*x*) exists and is given by

$$u(x) = e^{\int 1/x\,dx} = e^{\ln(x)} = x\ .$$

(4) The new equation is

$$(3x^2y + xy^2)dx + (x^3 + x^2y)dy = 0,$$

which is exact. (Check it!)

(5) Let us find $F(x,y)$. Consider the system:

$$\begin{cases} \frac{\partial F}{\partial x} = 3x^2y + xy^2, \\ \frac{\partial F}{\partial y} = x^3 + x^2y. \end{cases}$$

(6) Let us integrate the first equation. We get

$$F(x,y) = x^3y + \frac{x^2}{2}y^2 + \theta(y).$$

(7) Differentiate with respect to y and use the second equation of the system to get

$$\frac{\partial F}{\partial y} = x^3 + x^2y + \theta'(y) = x^3 + x^2y\ ,$$

which implies $\theta'(y) = 0$, that is, $\theta(y) = C$ is constant. Therefore, the function $F(x,y)$ is given by

$$F(x,y) = x^3y + \frac{x^2}{2}y^2\ .$$

We don't have to keep the constant C due to the nature of the solutions (see next step).

(8) All the solutions are given by the implicit equation

$$x^3y + \frac{x^2}{2}y^2 = C\ .$$

Remark: Note that if you consider the function

$$u(x,y) = \frac{1}{xy(2x+y)},$$

then we get another integrating factor for the same equation. That is, the new equation

$$\frac{1}{xy(2x+y)}(3xy+y^2)dx + \frac{1}{xy(2x+y)}(x^2+xy)dy = 0$$

is exact. So, from this example, we see that we may not have uniqueness of the integrating factor. Also, you may learn that if the integrating factor is given to you, the only thing you have to do is multiply your equation and check that the new one is exact.

5.9 SUMMARY

- We study $f(x,y)$ be a real valued function which is continuous on the rectangle $R = \{(x,y); |x - x_0| \le a\ ,\ |y - y_0| \le b\}$.

  Assume f has a partial derivative with respect to y and that $\frac{\partial f}{\partial y}$ is also continuous on the rectangle R. Then there exists an interval $I = [x_0 - h, x_0 + h]$ (with $h \le a$) such that the initial value problem

$$\begin{cases} y' = f(x,y)\ , \\ y(x_0) = y_0 \end{cases}$$

  has a unique solution $y(x)$ defined on the interval I.
- We also study (Cauchy-Peano Existence Theorem)

  Let $f : \mathbb{R}^2 \to \mathbb{R}$ b continuous in a neighborhood of the point $(x_0, y_0) \in \mathbb{R}^2$.
- We study tangent line to a function $f(x)$ at a given point x_0 is the best line approximating the function is to observe that the

tangent line is the (only) line passing through the point $(x_0, f(x_0))$ and having the same slope as *f*(*x*) at x_0

5.10 KEYWORD

Picard Iterative : Picard iteration for a differential equation is the process , which converges to a unique solution given that the function is continuous

Taylor Polynomial : A Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point. ... The polynomial formed by taking some initial terms of the Taylor series is called a Taylor polynomial

Integrating Factor : An integrating factor is any function that is used as a multiplier for another function in order to allow that function to be solved; that is, using an integrating factor allows a non-exact function to be exact.

5.11 EXERCISE

Q. 1 Find the 5th degree Taylor polynomial for the function $f(x) = \sin(3x)$ with center $x_0 = 0$.

Q. 2 FIND ALL THE SOLUTIONS TO

$$\frac{dy}{dx} = -\frac{3xy + y^2}{x^2 + xy}.$$

Q. 3 Find the quadratic Taylor polynomial for the function $f(x) = \frac{1}{x}$ with the center $x_0 = -2$.

Q. 4 Consider the autonomous differential equation with the initial condition

$$\frac{dy}{dt} = 18y^2, \quad y(0) = -0.1.$$

Find $\lim_{t \to +\infty} y(t)$.

Q. 5 Suppose the differential equation $\frac{dy}{dt} = f(t, y)$ satisfies the Existence and Uniqueness Theorem for all values of *y* and *t*.

Suppose $y_1(t) = t + 2e^{-t^2}$ and $y_2(t) = t$ are two solutions to this differential equation.

5.12 ANSWER TO CHECK IN PROGRESS

Check In Progress-1

Answer Q. 1 Check in section 4

Q. 2 Check in section 4

Check In progress-II

Answer Q. 1 Check in section 5

Q. 2 Check in section 7

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UNIT 6 GRONWALL'S INEQUALITY AND CONTINUATION OF SOLUTIONS

STRUCTURE

6.0 OBJECTIVES

- In This unit we study Gronwall's inequality and its proof with examples.
- We study Differential Inequality with its proof
- We also study theorem on *Uniqueness of Solutions to IVPs*
- We study integral invarint and its details

- We study continuation method (parametrized family) and also study continuation method (parametrized family, for non-linear operators)
- We study solvind of boundary value problems

6.1 INTRODUCTION

In mathematics, Grönwall's inequality (also called Grönwall's lemma or the Grönwall–Bellman inequality) allows one to bound a function that is known to satisfy a certain differential or integral inequality by the solution of the corresponding differential or integral equation. There are two forms of the lemma, a differential form and an integral form. For the latter there are several variants.

Grönwall's inequality is an important tool to obtain various estimates in the theory of ordinary and stochastic differential equations. In particular, it provides a comparison theorem that can be used to prove uniqueness of a solution to the initial value problem; see the Picard–Lindelöf theorem.

It is named for Thomas Hakon Grönwall (1877–1932). Grönwall is the Swedish spelling of his name, but he spelled his name as Gronwall in his scientific publications after emigrating to the United States.

The differential form was proven by Grönwall in 1919. The integral form was proven by Richard Bellman in 1943.

Suppose f is continuous. If x is a solution on an interval [a,a), a > a, we say x is a continuation of x if there is a b > a such that ~ is defined on [a-r,b), coincides with x on [a-r,a), and ~ satisfies on [a, b). A solution x is noncontinuable if no such continuation exists~ that is, the interval [a,a) is the maximal interval of existence of the solution x. The existence of a noncontinuable solution follows from Zorn's lemma. Also, the maximal interval of existence must be open.

6.2 GRONWALL’S INEQUALITY

Let $\sigma(n)$ be the divisor function. Then

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \ln \ln n} = e^{\gamma},$$

where γ is the Euler-Mascheroni constant. Ramanujan independently discovered a less precise version of this theorem

Theorem : Gronwall's Inequality

Let u, v be nonnegative continuous functions $[a, b]$ such that

$$v(t)C + \int_a^t v(s)u(s)ds,\ a \leq t \leq b,$$

then

$$v(t) \leq Ce^{\int_a^t u(s)ds}$$

In particular, if $C = 0$, then $v = 0$.

Proof. Let $h(t) := C + \int_a^t v(s)u(s)ds$ Therefore,

$$h'(t) = v(t)u(t) \leq h(t)u(t)$$

This reduces to the differential inequality

$$h' - uh \leq 0$$

Multiplying the LHS by

$$e^{-\int_a^t u(s)ds},$$

we get

$$\left(h(t)e^{-\int_a^t u(s)ds}\right)' \leq 0$$

And integrate from 0 to x to get

$$h(x)e^{-\int_a^x u(s)ds} - h(a) \leq 0$$

$$h(x) \leq h(a)e^{\int_a^x u ds}$$

Finally,

$$v(x) \leq h(x) \leq Ce^{\int_a^x u(s)ds}$$

This allows us to state a new uniqueness theorem:

Theorem : Uniqueness of Solutions to IVPs

Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on

$$Q := \{(x,y) : |x - x_0| \leq a, |y - y_0| \leq a\}$$

and satisfies

$$|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|.$$

Then the solution to the IVP (1) exists on $[x_0 - \alpha, x_0 + \alpha]$,

where $\alpha := \frac{a}{M}$, and the solution is unique.

Proof. Existence follows from

If there exists two solutions $\phi_1(t)$ and $\phi_2(t)$ to (1), then define

$$w(t) := \phi_1(t) - \phi_2(t)$$

Then, $w'(t) = \phi_1'(t) - \phi_2'(t)$, and

$$\int_{x_0}^{x} w'(t)dt = w(x) - w(x_0) = \int_{x_0}^{x} [f(t,\phi_1(t)) - f(t,\phi_2(t))]dt$$

$$w(x_0) = \phi_1(x_0) - \phi_2(x_0) = 0$$

So, we get the following for w:

$$w(x) = \int_{x_0}^{x} [f(t,\phi_1(t) - f(t,\phi_2(t))]dt$$

Therefore,

$$|w(x)| \leq \left| \int_{x_0}^{x} f(t,\phi_1) - f(t,\phi_2)dt \right|$$

$$\leq \int_{x_0}^{x} |f(t,\phi_1) - f(t,\phi_2)|dt \leq K \int_{x_0}^{x} |\phi_1(t) - \phi_2(t)|dt$$

$$= K \int_{x_0}^{x} |w(t)|dt$$

Thus, from Gronwell's Inequality with $u(t) := K$, $v(t) := |w(t)|$,

and $C = 0$, we get $|w(t)| = 0$. Thus, $\phi_1 = \phi_2$, and the uniqueness is shown. Λ

Notes

6.3 DIFFERENTIAL INEQUALITY

An inequality which interconnects the argument, the unknown function and its derivatives, e.g.

$$y'(x) > f(x, y(x)),$$

where y is an unknown function of the argument x. The principal problem in the theory of differential inequalities is to describe, starting from a known differential inequality and additional (initial or boundary) conditions, all its solutions.

Differential inequalities obtained from differential equations by replacing the equality sign by the inequality sign — which is equivalent to adding some non-specified function of definite sign to one of the sides of the equation — form a large class. A comparison of the solutions of such inequalities with the solutions of the corresponding differential equations is of interest. Thus, the following estimates [1] are valid for any solution of (1):

$$y(x) < z(x) \quad \text{if } x_1 \leq x < x_0,$$
$$y(x) > z(x) \quad \text{if } x_0 < x \leq x_2,$$

where

$$z' = f(x, z), \qquad z(x_0) = y(x_0),$$

on any interval $[x_1, x_2]$ of existence of both solutions. This simple statement is extensively employed in estimating the solutions of differential equations (by passing to the respective differential inequality with a particular solution which is readily found), the domain of extendability of solutions, the difference between two solutions, in deriving conditions for the uniqueness of a solution, etc. A similar theorem [2] is also valid for a differential inequality (Chaplygin's inequality) of the type

$$y^{(m)} + a_1(x)y^{(m-1)} + \ldots + a_m(x)y > f(x).$$

Here, estimates of the type (2) for solutions satisfying identical initial conditions at $x = x_0$ are only certainly true on some interval determined by the coefficients $a_1, \ldots, a_m$. E.g., this is the interval $[x_0 - \pi, x_0 + \pi]$ for $y'' + y > f$.

For a system of differential inequalities

$$y_i'(x) > f_i(x, y_1, \dots, y_n), \qquad i = 1, \dots, n,$$

it has been shown [3] that if each function f_i is non-decreasing with respect to the arguments y_j (for all $j \neq i$), the estimate

$$y_i(x) > z_i(x) \quad \text{if } x_0 < x \leq x_2; \qquad i = 1, \dots, n,$$

resembling (2), is valid. The development of these considerations leads to the theory of differential inequalities in spaces with a cone.

A variant of differential inequalities is the requirement that the total derivative of a given function is of constant sign:

$$\frac{d}{dx} F(x, y_1, \dots, y_n) \equiv \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y_1} y_1' + \dots + \frac{\partial F}{\partial y_n} y_n' \leq 0 .$$

This requirement is used in stability theory.

A representative of another class is the differential inequality

$$\max_{i=1,\dots,n} |y_i' - f_i(x, y_1, \dots, y_n)| \leq \epsilon$$

($\epsilon > 0$ is given), which was first studied in the context of the general idea of an approximate description of a real problem by differential equations. Here the description of the integral funnel, i.e. the set of all points of all solutions which satisfy the given initial conditions, in particular, the behaviour of the funnel as $x \to \infty$, is of interest. A natural generalization of the differential inequality (3) is a differential equation in contingencies, specified by a field of cones, which generalizes the concept of a field of directions.

The theory of boundary value problems was also studied for differential inequalities. The inequality $\Delta u \geq 0$, where Δ is the Laplace operator, defines subharmonic functions; the differential inequality $\partial u / \partial t - \Delta u \leq 0$ defines subparabolic functions. Studies were also made of differential inequalities of a more general type (in both the above classes) with partial derivatives for differential operators of various types.

6.4 INTEGRAL INVARIANT

Integral Invariant of degree (order) k, of a smooth dynamical system

An absolute integral invariant is an exterior differential form ϕ of degree k that is transformed into itself by the transformations generated by this system.

A relative integral invariant is an exterior differential form ϕ of degree k whose exterior differential is an absolute integral invariant (of degree $k+1$).

One usually speaks of integral invariants of a flow (continuous-time dynamical system) $\{S_t\}$ defined by a system of ordinary linear differential equations $\dot{x} = f(x)$, where f is a smooth vector field given on some domain in a Euclidean space (or on a manifold). In coordinates (local coordinates in the case of a manifold) this system has the form

$$\dot{x}_i = f_i(x_1, ..., x_n), \qquad i = 1, ..., n.$$

An important example of an integral invariant is a volume form $\phi(x) = \rho(x)\, dx_1 \wedge \ldots \wedge dx_n$ (where $\rho(x)$ is a positive locally integrable (often even continuous or smooth) function in the coordinates). For smooth ρ this form is an absolute invariant of (1) if

$$\text{div}\,(\rho f) = \sum_{i=1}^{n} \frac{\partial(\rho f_i)}{\partial x_i} = 0.$$

In this case the flow has an invariant measure $\mu(A) = \int_A \phi$, which is given in (local) coordinates by its density $\rho(x)$ (the latter is often called an integral invariant, allowing for some ambiguity of speech).

A Hamiltonian system with (generalized) momenta and coordinates p_i, q_i, $i = 1, ..., m$, has the relative integral invariant

$$\psi = \sum p_i\, dq_i$$

and the absolute integral invariant

$$\omega = \sum dp_i \bigwedge dq_i.$$

This fact may be put at the basis of the definition of a Hamiltonian system and may be used to develop the theory of Hamiltonian systems, since many specific properties of such systems are directly related to these integral invariant. The exterior powers ω^k (including the volume form ω^m) are absolute, while the products $\psi \wedge \omega^k$ are relative integral invariants of any Hamiltonian system. Therefore, they are called universal integral invariants of Hamiltonian systems. Up to a multiplier, all universal integral invariants of Hamiltonian systems can be reduced to the ones indicated.

If (1) has an absolute integral invariant ϕ of degree k, then for any k-dimensional smooth chain c (e.g., for a smooth k-dimensional manifold),

$$\int_c \phi = \int_{S_t(c)} \phi \ .$$

If (1) has a relative integral invariant, then (2) holds, generally speaking, only when the chain is the boundary of a chain of dimension $k+1$. Sometimes relative integral invariants are defined by the stronger condition that (2) holds for all cycles c. Initially, integral invariants were defined by H. Poincaré as integrals of the type above that remain invariant under the action of the flow on the domain of integration.

All that has been said can easily be generalized to non-autonomous systems $\dot{x} = f(x, t)$. The modification given by E. Cartan appears to be most essential. It involves the transition (even in the autonomous case) to an extended phase space (one adds time to the ordinary phase coordinates), in which the integral curves (cf. Integral curve) of the system of differential equations considered form a certain family of lines (a congruence, cf. Congruence of lines). Cartan requires that the integral of a form ϕ over a chain c (or over a cycle, if one discusses a relative integral invariant) remains invariant if each point $(x, t) \in c$ moves along the integral curve passing through this point; different points may move in different ways as long as this gives a smooth deformation of c. (E.g., in the new sense it is not ψ that is a relative integral invariant of Hamiltonian systems, but the extremely useful Cartan–Poincaré integral invariant $\sum p_i dq_i - H dt$, where H is the Hamiltonian.

6.5 CONTINUATION METHOD (PARAMETRIZED FAMILY)

The inclusion of a given problem in a one-parameter $(0 \leq \alpha \leq 1)$ family of problems, connecting the given problem $(\alpha = 1)$ with a problem that is known to be solvable $(\alpha = 0)$, and the study of the dependence of solutions on the parameter α. The method is extensively used in the theory of differential equations.

Suppose, e.g., that one has to prove the solvability in a Hölder class of the Dirichlet problem

$$\left.\begin{aligned} &Lu = f, \\ &u|_{\partial D} = \phi, \end{aligned}\right\}$$

in a suitable bounded N-dimensional region D for the linear elliptic second-order operator

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

$$c(x) \leq 0; \quad a_{ij}, b_i \in C^{(n-2,\,\mu)}(\overline{D}); \quad \overline{D} = D \bigcup \partial D;$$

$$n \geq 2; \quad \mu > 0 .$$

One introduces the family of elliptic operators

$$L_\alpha u = \alpha Lu + (1 - \alpha)\Delta u, \quad 0 \leq \alpha \leq 1,$$

and considers for it the Dirichlet problem

$$\left.\begin{aligned} &L_\alpha u = f \quad \text{in } D, \\ &u|_{\partial D} = \phi . \end{aligned}\right\}$$

Let $\mathfrak{A}$ be the set of all $\alpha \in [0, 1]$ for which (2) is uniquely solvable in $C^{(n,\,\mu)}(\overline{D})$ for any $f \in C^{(n-2,\,\mu)}(\overline{D})$ and $\phi \in C^{(n,\,\mu)}(\partial D)$. The set $\mathfrak{A}$ is not empty, since for $\alpha = 0$ (i.e. for the Laplace operator) (2) is uniquely solvable in $C^{(n,\,\mu)}(\overline{D})$, which follows from potential theory. The set $\mathfrak{A}$ is at the same time open and closed in $[0, 1]$, hence coincides with it. Thus, $\alpha = 1$ belongs to $\mathfrak{A}$ and (1) is solvable.

The continuation method (in the case of analytic continuation) was proposed and developed in a number of papers by S.N. Bernstein [S.N. Bernshtein], . Subsequently, this method gained wide application in various problems in the theory of linear and non-linear differential equations, where the idea of analytic continuation was supplemented by more general functional and topological principles.

Check In Progress-I

Q. 1 State Gronwall's Inequality.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q. 2 Define differential Inequality.

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

6.5.1 Continuation Method (Parametrized family, for Non-Linear Operators)

A method for approximately solving non-linear operator equations. It consists of generalizing the equation to be solved, $P(x) = 0$, to the form $F(x, t) = 0$, by introducing a parameter t that takes values in a finite interval, $t_0 \leq t \leq t^*$, such that the initial equation is obtained

Notes

for $t=t^*$: $F(x, t^*)=P(x)$, while the equation $F(x, t_0)=0$ can either easily be solved, or a solution x_0 of it is already known.

The generalized equation $F(x, t)=0$ is solved sequentially for individual values of t: $t_0, \dots, t_k=t^*$. For $t=t_{i+1}$ it is solved by means of some iteration method (Newton, simple iteration, variation of parameter, etc.), starting with the solution x_i obtained by solving $F(x, t)=0$ for $t=t_i$. Applying at each step in i, e.g., n Newton iterations, leads to the formulas

$$x_{i+1}^{(v+1)}=x_i^{(v)}-\left[F_x'(x_{i+1}^{(v)}, t_{i+1})\right]^{-1}F(x_{i+1}^{(v)}, t_{i+1}),$$

$$i=0, \dots, k-1; \qquad v=0, \dots, n-1; \qquad x_{i+1}^{(0)}=x_i^{(n)}.$$

If the difference $t_{i+1}-t_i$ is sufficiently small, then the value of x_i may turn out to be a sufficiently good initial approximation, ensuring convergence, in order to obtain the solution x_{i+1} for $t=t_{i+1}$ (cf.

In practice, the initial problem often naturally depends on some parameter, which can then be taken as t.

The continuation method is used in the solution of systems of non-linear algebraic and transcendental equations, as well as for more general non-linear functional equations in Banach space.

The continuation method is sometimes called the direct method of variation of parameter as well as the combined method of direct and iterative variation of parameter. In these methods the construction of solutions of generalized equations is reduced, by differentiation with respect to the parameter, to the solution of a differential problem with initial conditions (a Cauchy problem) by methods of numerical integration of ordinary differential equations. Applying the simplest Euler method in the direct method of variation of parameter to the Cauchy problem

$$\frac{dx}{dt}=-\left[F_x'(x, t)\right]^{-1}F_t'(x, t), \qquad x(t_0)=x_0,$$

the approximate values $x(t_i)=x_i$, $i=1, \dots, k$, of the solution $x(t)$ of $F(x, t)=0$ can be determined from the following identities:

$$x_{i+1} = x_i - (t_{i+1} - t_i)\left[F'_x(x_i, t_i)\right]^{-1} F'_t(x_i, t_i),$$
$$i = 0, ..., k-1.$$

The element x_k is the required approximate solution of the initial equation $P(x) = 0$. A refinement of all or some values x_{i+1} can be obtained by the iteration method of variation of parameter (or Newton's method). The generalized equation is here usually generated in the form

$$F(x, t_{i+1}) = (1 - \lambda)F(x^{(0)}, t_{i+1}), \qquad x^{(0)} = x_{i+1},$$

on a finite interval $0 \leq \lambda \leq 1$, or, replacing in it $1 - \lambda$ by $e^{-\tau}$, on the infinite interval $0 \leq \tau \leq \infty$.

The method of variation of parameter has been applied to large classes of problems both for constructing solutions, as well as for proving their existence

6.6 ASCOLI-ARZELA THEORY

We aim to state the Ascoli-Arzela Theorem in a bit more generality than in previous classes.

Definition 7.1 Complete Metric Space

Let X be a metric space. Then X is said to be complete if every Cauchy sequence in X converges to a point in X.

In other words, if $\{x_n\} \subset X$, and $d(x_n, x_m) \to 0$ as $n, m \to \infty$, then there exists $\hat{x} \in X$ with $\lim_{n\to\infty} x_n = \hat{x}$.

Definition 7.2 ϵ-Ball

For $x \in X$ and $\epsilon > 0$, we define the ϵ-ball about x to be

$$B(x; \epsilon) := \{y \in X : d(x, y) < \epsilon\}.$$

Definition 7.3 ϵ-Nets

Let $S \subset X$ is such that

$$X = \bigcup_{x \in S} B(x; \epsilon)$$

then the set S is called an ϵ-net in X (for X). In other words, every point in X is within ϵ of some point in S.

Notes

Definition 7.4 Totally Bounded We say that the metrix space X is totally bounded (or precompact) if for any $\epsilon > 0$, there is a finite ϵ-net for X.

Recall that if X is a compact metric space, then X is totally bounded. That is, we have for any $\epsilon > 0$

$$X = \bigcup_{x \in X} B(x; \epsilon)$$

so $\{B(x; \epsilon)\}_{x \in X}$ is an open cover, so there is a finite subcover

$$B(x_1; \epsilon), \ldots, B(x_n; \epsilon)$$

So that

$$X = \bigcup_{i=1}^{n} B(x_i; \epsilon).$$

Equivalently, $x_1, \ldots, x_n$ is a ϵ-net for X.

Theorem 7.1 If X is a complete and totally bounded metric space, then X is compact.

Proof. If no, then there exists an open cover $\{U_\alpha\}$ of X which does not have a finite subcover. We use this to define a sequence in X as follows:

Let $\{y_1, \ldots, y_n\}$ be a finite 1-net for X. It follows that, for some j, $B(y_j, 1)$ cannot be covered by a finite subcover of $\{U_\alpha\}$. We let $x_1 = y_{j_0}$. In a similar manner, we pick x_i for a finite $\frac{1}{i}$-net of $B(x_{i-1}, \frac{1}{i-1})$. We know that the sequence $\{x_n\}$ is a Cauchy sequence, and much converge to some $\hat{x}$. However, there is a $U' \in \{U_\alpha\}$ which contains $\hat{x}$. So, by picking a sufficiently large enough N, we can get that the ball $B(x_N, \frac{1}{N}) \subset U'$, a contradiction.

Proposition 7.1 If $S \subset X$ is totally bounded, where X is is a complete metric space, then $\bar{S}$ is also totally bounded.

Proof. Let $\epsilon > 0$ be given. Claim that $\bar{S}$ has a finite ϵ-net; (i.e., there exists $x_1, \ldots, x_n \subset \bar{S}$ such that $\bigcup_1^n B(x_i; \epsilon) = X$.

To see this, let $\{x_1, \ldots, x_n\}$ be a finite $\frac{\epsilon}{2}$-net for S. Then, for $y \in S$, $y \in B(x_i; \frac{\epsilon}{2})$ for some i. If $\hat{y} \in \bar{S}$, then there exists a sequence $\{y_m\} \subset S$ with $\lim y_m = \hat{y}$. Therefore, we find a k such that $d(\hat{y}, y_k) < \frac{\epsilon}{2}$ and an i_0 such that $y_k \in B(x_{i_0}, \frac{\epsilon}{2})$. Thus, by the triangle inequality, we get that $d(\hat{y}, x_{i_0}) < \frac{\epsilon}{2}$. Thus, $\{x_i\}$ is an ϵ-net for $\bar{S}$. Λ

Definition 7.5 $B(X)$, $C(X)$**:**

If X is a metrix space, then

$$B(X) := \{u : X \to \mathbb{R} | u \text{ bounded}\}$$

$$C(X) := \{u : X \to \mathbb{R} | u \text{ is continuous on } X\}$$

For $u, v \in B(X)$,

$$d(u, v) := \sup_X |u(x) - v(x)| = \|u - v\|$$

Convergence in this norm is uniform convergence. We use the same norm for $C(X)$.

Both $C(X)$ and $B(X)$ are complete metric spaces.

Definition 7.6 Bounded:

If $A \subset C(X)$, then A is said to be bounded if there exists a $M > 0$ such that for all $u \in A$, $\|u\| \leq M$.

Definition 7.7 Relatively Compact:

If $A \subset X$, where X is a compact metric space, then we say A is relatively compact if it is contained in some compact subset of X.

Definition 7.8 Equicontinuous:

If $A \subset C(X)$, then we say that A is equicontinuous at $x \in X$ if for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x)$ such that for some $y \in X$ where $d(y, x) \subset \delta$ implies $|u(y) - u(x)| < \epsilon$ for all $u \in A$.

A is said to be equicontinuous on X if it is equicontinuous at all $x \in X$.

Theorem 7.2 Ascoli-Arzela Theorem:

Let X be a compact metric space and let $A \subset C(X)$ be equicontinuous and bounded. Then, A is relatively compact.

Proof. Let $\epsilon > 0$ be given. Then, for each $x \in X$, $\exists \delta_x > 0$ such that $d(x, y) < \delta_x$ implies $|u(x) - uu(y)| < \frac{\epsilon}{3}$ for all $u \in A$.

Therefore, $X \subset \bigcup_{x \in X} B(x; \delta_x)$ (open cover). As X is compact, there is a finite subset $\{x_1, \ldots, x_n\} \subset X$ such that $X \subset \bigcup_{i=1}^{n} B(x_i, \delta_{x_i})$. (From here on, denote $\delta_i := \delta_{x_i}$). We need to show that A is totally bounded - i.e., we need to find a finite ϵ-net for A. For each j, $1 \leq j \leq n$, the set

$$A_j := \{u(x_j) : u \in A\}$$

is a bounded set in $\mathbb{R}$. Therefore, A_j has a finite $\frac{\epsilon}{3}$-net (in $\mathbb{R}$). That is, $\exists u_i^j, 1 \leq i \leq k(j)$ so that $\{u_i^j(x_j)\}_{i=1}^{k(j)}$ is a $\frac{\epsilon}{3}$-net in $\mathbb{R}$, i.e. for any $u \in A$, $|u(x_j) - u_i^j(x_j)| \leq \frac{\epsilon}{3}$ for some $1 \leq i \leq k(j)$.

We claim that $\{u_i^j\}$ is an ϵ-net for A. For any $\hat{u} \in A$ and $x \in X$, then x is within δ_{j_0} of some x_{j_0}. So, $|u(x) - u(x_{j_0})| < \frac{\epsilon}{3}$ by equicontinuity. Now $\hat{u}(x_{j_0})$ is within $\frac{\epsilon}{3}$ of some $u_{i_0}^{j_0}(x_{j_0})$ by the $\frac{\epsilon}{3}$-net property of the $A_{j_0} \subset \mathbb{R}$. Therefore, $|\hat{u}(x_{j_0}) - u_{i_0}^{j_0}(x_{j_0})| < \frac{\epsilon}{3}$. Then, for $\hat{u} \in A$ and x as above,

$$|\hat{u}(x) - u_{i_0}^{j_0}(x)| \leq |\hat{u}(x) - \hat{u}(x_{j_0})| + |\hat{u}(x_{j_0}) - u_{i_0}^{j_0}(x_{j_0})| + |u_{i_0}^{j_0}(x_{j_0}) - u_{i_0}^{j_0}(x_{j_0})|$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Theorem 7.3 Let X be a compact metric space. Then every bounded equicontinuous family has a uniformly convergent subsequence.

Definition 7.9 Compact Mapping:

Let X, Y be complete metric spaces and $f : X \to Y$ be continuous. Then $f : X \to Y$ is said to be a compact mapping if $f(X)$ is relatively compact in Y.

Definition 7.10 Complete Continuity:

We say $f : X \to Y$ is completely continuous if for any bounded set $S \subset X$, $f(S)$ is relatively compact.

Recall that a normed linear space is a vector space V wih a mapping $\|\cdot\| : V \to \mathbb{R}^+$ such that

- $x \in V \implies \|x\| \geq 0$, and equality holds iff $x = 0$.
- $\|cx\| = |c|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Theorem 7.4 Compositions of Completely Continuous Funcions:

Let X, Y, and Z be normed linear spaces and suppose $f : X \to Y$ is completely continuous and $g : Y \to Z$ is continuous.

Then, $g \circ f : X \to Z$ is completely continuous.

Proof. Let $S \subset X$ be bounded and consider a sequence $z_n \in g(f(s))$; thus, $z_n = g(f(x_n)), x_n \in X$. We claim some subsequence of z_n converges to $\hat{z} \in Z$ (note that $\hat{z}$ need not be in $g(f(S))$.

Note that there is a subsequence $\{z_{n_k}\}$ such that $f(z_{n_k}) \to \hat{y} \in Y$.

Further, $g : Y \to Z$ is continuous, so $\lim_{n\to\infty} g(f(z_{n_k})) = g(\hat{y}) \in Z$.

Thus, $g \circ f : X \to Z$ is completely continuous.

Notation: If $I == [a, b]$ then $C^k(I)$ denotes the space of all real-valued continuous functions $u : I \to \mathbb{R}$ such that $u^{(j)}$ is continuous on $[a, b]$ for $j = 0 \ldots k$.

$C^K(I)$ is a normed linear space with norm

$$\|u\|_k = \sum_{j=0}^{k} \|u^{(j)}\|$$

where $\|u^{(j)}\| = \sup_{t\in I} |u^{(j)}(t)|$.

Clearly, $u \in C^k \implies u \in C^{k-1}$ and the embedding $C^{k+1} \to C^k$ is completely coninuous.

Theorem 7.5 $\tau : C^{k+1} \to C^k$ is completely continuous, where

$$\tau(u) = u$$

Proof. $C^1 \subset C$ is a completely continuous embedding.

If $S \subset C^1$ and S is bounded, then we claim that S is relatively compact in C. That is, if $\{u_n\} \subset S \subset C^1$ and $\|u_n\|_1 \leq M \forall n$, then $\|u_n'\| \leq M$. Give $\epsilon >$ and let $\delta = \frac{\epsilon}{M}$. Then, by the MVT, for any $x, y \in I$, $|x - y| < d$ and $|u_n(x) - u_n(y)| \leq M|x - y| < \epsilon$ Therefore, $\{u_n\}$ is a bounded equicontinuous family and so contains a convergent subsequence by the Arzela Ascoli theorem. Λ

6.6.1 Solving BVPs

We are next interested in applying these theorems in the BVP

$$y'' = f(t, y, y')$$

$$y(0) = y(1) = 0$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. This includes, for example, the forced pendulum equation in the form

$$y'' + a \sin y = e(t)$$

$$y(0) = y(\frac{T}{2}) = 0$$

where $e(t)$ is t-periodic and odd. In this case, we take

$$f(t, u, v) = \frac{T^2}{4}(e(\frac{T}{2}t) - a \sin u),$$

where we have done the change of variable

$$t = \frac{T}{2}s;\ y(t) = u(s).$$

Write

$$Ly = y''.$$ (linear differential operator)

Then, $L : C^2[0,1] \to C[0,1]$. We introduce the *Nemitski Operator*, $F : C^1[0,1] \to C[0,1]$ given by

$$F(u)(t) = f(t, u(t), u'(t)).$$

Notes

Let j be the inclusion operator from C^2 to C^1. To solve the boundary value problem, we need to invert L and apply it to $Ly = f(t, y, y')$.

First, we need to restrict L to the subspace

$$C_0^2[0,1] = C_0^2 = \{u \in C^2 : u(0) = u(1) = 0\};$$

then we will have the solution to the BVP written as a solution to the operator equation,

$$y = (L^{-1} \circ F \circ j)(y)$$

Define $S := L^{-1} \circ F \circ j : C_0^2 \to C_0^2$.

Recall the Green's function for the problem

$$u'' = 0,\ u(0) = u(1) = 0,$$

given by

$$G(s,t) := \begin{cases} (t-1)s, & 0 \le s \le t \le 1 \\ t(s-1), & 0 \le t \le s \le 1 \end{cases}$$

Recall,

Lemma 7.1 *For* $w \in C$, $u = L^{-1}w$ *is given by*

$$u(t) = \int_0^1 G(s,t)w(s)ds$$

and satisfies

$$u''(t) = w,\ u(0) = u(1) = 0.$$

Proof. Verify this directly (we've seen it before).

We also have,

Lemma 7.2 *For* $w \in C$,

$$\|L^{-1}w\|_2 \le \frac{7}{4}\|w\|$$

Proof. Recall for $u \in C^2$, the norm

$$\|u\|_2 = \max|u(t)| + \max|u'(t)| + \max|u''(t)|$$

$$= \|u\| + \|u'\| + \|u''\|$$

Notice that

$$G(s,t) \leq 0$$

and it is easy to see that

$$-\frac{1}{4} \leq G(s,t) \leq 0.$$

So letting $u = L^{-1}w$,

$$|u(t)| = |L^{-1}w(t)| \leq \int_0^1 |G(t,s)||w(s)|ds \leq \frac{1}{4}\|w\|.$$

Now,

$$u'(t) = \int_0^t sw(s)ds + \int_t^1 (s-1)w(s)ds$$

$$|u'(t)| = \left|\int_0^t sw(s)ds + \int_t^1 (s-1)w(s)ds\right|$$

$$\leq \|w\| \left(\int_0^t sds + \int_t^1 (s-1)ds\right)$$

$$\leq \|w\| \left(\frac{t^2}{2} + \frac{(1-t)^2}{2}\right) \leq \frac{1}{2}\|w\|$$

Finally, through similar estimates, we get $\|u''(t)\| = \|w(t)\| \leq \|w\|$, so

$$\|L^{-1}w\|_2 = \|u\|_2 \leq \left(\frac{1}{4} + \frac{1}{2} + 1\right)\|w\|$$

$$= \frac{7}{4}\|w\|$$

Hence, as L^{-1} has this uniform bound, we get that it is continuous (from a simple proof in functional analysis). Λ

Monday, 2-14-2005

Lemma 7.3 *If* $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ *is continuous, then the Nemitski operator* $F:C^1[0,1]\to C[0,1]$ *is continuous in* C^1*, where*

$$(Fu)(t):=f(t,u(t),u'(t))$$

Proof. Let $u\in C^1[0,1]$, and let $\epsilon>0$ be given. We need to show there exists $\delta=\delta(\epsilon,u)>0$ such that if $v\in C^1[0,1]$ and $\|u-v\|_1<\delta$, then $\|F(v)-F(u)\|<\epsilon$. Chose $r\geq 1$ such that $\|u\|_1\leq r$. Now, on the compact set

$$Q:=[0,1]\times[-2r,2r]\times[-2r,2r],$$

f is bounded and uniformly continuous. So, there exists a $\hat{\delta}>0$ such that if $|t_1-t_2|+|u_1-u_2|+|v_1-v_2|<\hat{\delta}$, then

$$|f(t,u_1,v_1)-f(t,u_1,v_1)|<\epsilon,\ \forall(t_i,u_i,v_i)\in Q,\ i=1,2$$

Let $\delta=\min\{\hat{\delta},r\}$. If $v\in C^1[0,1]$ and $\|u0v\|_1<\delta$, then

$$\|v\|_1\leq\|v-u\|_1+\|u\|_1<\delta+r\leq 2r.$$

Hence,

$$|v(t)|<2r, \quad \text{and} \quad |v'(t)|<2r$$

So, $(t,u(t),u'(t))\in Q$ and $(t,v(t),v'(t))\in Q$.

Hence, $|f(t,u(t),u'(t))-f(t,v(t),v'(t))|<\epsilon$ for all t and $\|F(u)-F(v)\|<\epsilon$. Thus, $F:C^1[0,1]\to C[0,1]$ is continuous.

Λ

We now have:

Theorem 7.1.1 The operator

$$S:C_0^2[0,1]\to C_0^2[0,1],$$

given by,

$$(S(y))(t) = L^{-1}(F(j(y))(t)$$

is completely continuous.

Proof. Immediate. $j : C^2[0,1] \to C^1[0,1]$ given by $j(u) = u$ is completely continuous. Then S is also, as it is the composition of a continuous and completely continuous map. Λ

We now easily prove:

Theorem 7.1.2 If $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is bounded, then the BVP

$$y'' = f(t, y, y')$$

$$y(0) = y(1) = 0$$

has a solution.

Proof. Let

$$m := \sup\{|f(t,u,v)| : (t,u,v) \in [0,1] \times \mathbb{R}^2\};$$

$$B := \{u \in C_0^2[0,1] : \|u\|_2 \leq \frac{7}{4}m\}.$$

where

$$\|u\|_2 = \max|u(t)| + \max|u'(t)| + \max|u''(t)|.$$

Then we note that B is closed, bounded, and convex. We claim $S(B) \subset B$. Let $u \in B$; then we want to show that $S(u) \in B$.

We note that

$$\|F(j(u))\| \leq m.$$

i.e.,

$$|f(t, u(t), u'(t)) \leq m, \forall t \in [0,1].$$

Let $S = L^{-1} \circ F \circ j$. Then,

$$\|Su\|_2 = \|L^{-1}(F(j(u)))\|_2 \leq \frac{7}{4}\|F(j(u))\| \leq \frac{7}{4}m$$

Thus,

$$S(u) \in B$$

and S is a completely continuous operator in B into B, and so has a fixed point $u \in B$ with $S(u) = u = L^{-1}(F(j(u)))$. Then,

$$Lu = F(j(u)) = f(u, u, u')$$

$$u(0) = u(1) = 0$$

and the BVP has a solution. Λ

We get this as an immediate Corollary:

Theorem 7.1.3 Suppose $e : \mathbb{R} \to \mathbb{R}$ is an odd T-periodic function . Then the BVP

$$y'' = a \sin y = e,$$

$$y(0) = y(\frac{T}{2}) = 0,$$

has a solution. By extending y to an odd function and extending, we can find a periodic solution of the BVP for all of $\mathbb{R}$.

Check In Progress-II

Q. 1 Define Continuation Method .

Solution : . . . .
. . . .
. . . .
. . . .
. . . .
. . . .

Q. 2 *For* $w \in C$, $u = L^{-1}w$ *is given by*

$$u(t) = \int_0^1 G(s,t)w(s)ds$$

and satisfies

$$u''(t) = w,\ u(0) = u(1) = 0.$$

Solution : . . . .

. . . .

. . . .

. . . .

. . . .

. . . .

6.6.2 BVPs for Non-Bounded f

Remarks: (on steady-state temperature). Suppose we have a rod of finite length whose ends are kept at a fixed temperature. Under certain simplifying asumptions, the temperature at any point $s \in [0,1]$ satisfies a certain second order differential equation of the form

$$ky'' + k'(y')^2 + q(s,y) = 0, 0 < s < 1,$$

where k is the thermal conductivity depending on $y = y(s)$. In any case, this can be written as

$$y'' = f(s,y,y'), 0 < s < 1$$

$$y(0) = y(1) = 0$$

where $f(s,u,p)$ has the form

$$f(s,u,p) = -\frac{1}{k(u)}[k'(u))p^2 + q(s,u)]$$

The basic assumptions on f are that $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exists an $M > 0$ such that if

1. $|u| > M$ implies $uf(s,u,0) > 0$, for all $0 \le s \le 1$
2. There exists $A, B > 0$ such that if $0 \le s \le 1$, $|u| < M$, then

$$|f(s,u,p)| < Ap^2 + B, \forall p,$$

i.e., f grows no faster than quadratically with respect to p for s, u bounded.

Note: Topics like this lead into generalized Bernstein Theory, where we are interested in the BVP

$$y'' = f(s, y, y')$$

$$y(0) = y(1) = 0$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous which satisfies the above conditions. Again, we will consider the map

$$S : C^2[0,1] \to C^2[0,1]$$

defined by $S(u) := L^{-1}(F(j(u)))$ as before. Note that F no longer has a bounded image; however, we will still be able to show that S is completely continuous and apply the Leray-Schauder alternative.
To apply the Leray-Schauder Alternative, we need to show that there exists a $r > 0$ such that if $\|u\|_2 = r$, $u \in C_0^2[0,1]$ then $Su \neq \lambda u$ for all $\lambda > 1$. We will show that if $Su = \lambda$ for some $\lambda > 1$ where $u \in C_0^2[0,1]$, then

$$\|u\|_2 < r.$$

Consider the equation $Su = \lambda u$. Then, $Su = L^{-1}(F(j(u))) = \lambda u$.
Then,

$$F(j(u)) = L(\lambda u) = \lambda u''$$

and

$$u'' = \frac{1}{\lambda} f(t, u, u'), \; u(0) = u(1) = 0.$$

We write this as

$$u'' = f_\lambda(t, u, u')$$

where $f_\lambda = \frac{1}{\lambda} f$. Therefore, we can show that if such a problem has a solution, then $\|u\|_2, r$ (for some r).

We state and prove:

Theorem 7.2.1 *Let* $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous and satisfy the growth conditions stated previously. Then there exists a* $r > 0$ *such that if* $y = y(s)$ *is a solution of*

$$y'' = f_\lambda(s, y, y')$$

$$y(0) = y(1) = 0$$

for $\lambda > 1$*, then* $\|y\|_2 < r$.

Remark: Notice that if u is a solution of

$$u'' = f_\lambda(s, u, u'),\ u(0) = u(1) = 0$$

and $|u| > M$, then $u(s_0) = \max_{s \in [0,1]} u(s)$, $u'(s_0) = 0$, and $u''(s_0) \leq 0$. However, by one of the conditions that we are assuming on f, $u''(s_0) = f_\lambda(s_0, u(s_0), 0) > 0$.

Proof. Recall that $\|u\|_2 = \|u\| + \|u'\| + \|u''\| = \max|u(s)| + \max|u'(s)| + \max|u''(s)|$. Define:

$$r(s) = \frac{1}{2}(y(s))^2,$$

where y solves the BVP

$$y'' = f_\lambda(s, y, y')$$

$$y(0) = y(1) = 1.$$

We claim that $\|y\| \leq M$. TO see this, note that $r(0) = r(1) = 0$.

Since $r \geq 0$, there exists an $s_0 \in (0,1)$ where $r(s_0) = \max r(s)$.

Thus, $r(s_0) = 0$ and $r''(s_0) \leq 0$ by the second derivative test.

Then, $r'(s) = y(s)y'(s)$ and $r''(s) = y(s)y''(s) + (y'(s))^2$. Therefore at $s = s_0$, $r'(s_0) = 0$ implies $y'(s_0) = 0$ and

$$r''(s_0) = y(s_0)y''(s_0) = y(s_0)f_\lambda(s_0, y(s_0), 0) \leq 0.$$

It follows that $|y(s_0)| \leq M$ (see the above remark) and that $\|y\| \leq M$. So, for u, we can apply the second condition - the growth condition on f. We will use this to find an a priori bound on $\|y'\|$.

We claim that for all $s \in [0,1]$, we have $|y'(s)| \leq M_1$ where

$$M_1 := \sqrt{\frac{B}{A}(e^{4AM} - 1)}.$$

There are four cases to consider, as we know that there exists $\hat{s} \in [0,]$ with $y'(\hat{s}) = 0$:

1. $y'(s) > 0$ on (s_0, s_1), $y'(s_1) = 0$
2. $y'(s) < 0$ on (s_0, s_1), $y'(s_1) = 0$
3. $y'(s) > 0$ on (s_0, s_1), $y'(s_0) = 0$
4. $y'(s) < 0$ on (s_0, s_1), $y'(s_0) = 0$

We can divide the interval $[0,1]$ into a finite number of such subintervals. Assume that 1 holds (the other cases are similar) so

$$y'(s) > 0 \quad \text{on} \quad (s_0, s_1)$$

$$y'(s_1) = 0.$$

Because $\|y\| \leq M$,

$$|f(s,u,p)| \leq Ap^2 + B$$

for all $0 \leq s \leq 1$, $\|u\| \leq M$, and all p.

Note:

$$y''(s) = f_\lambda(s,y,y') = \frac{1}{\lambda} f(s,y,y')$$

$$-y''(s) \leq |y''(s)| \leq \frac{1}{\lambda}|f(s,y,y')| \leq |f(s,y,y')|$$

as $\lambda > 1$. Then,

$$-y''(s) \leq A(y')^2 + B$$

Multiply by $-2Ay' <$ on (s_0, s_1) and get

$$\frac{2Ay'y''}{A(y')^2 + B} \geq -2Ay'$$

Integrating from s_0 to s_1,

$$\int_{s_0}^{s_1} \frac{2Ay'y''}{A(y')^2 + B} \geq -\int_{s_0}^{s_1} 2Ay'.$$

Defining $V = A(y')^2 + B$,

$$\int_{V(s_0)}^{V(s_1)} \frac{dV}{V} \geq -2A(y(s_1) - y(s_0))$$

$$\ln V(s_1) - \ln V(s_0) \geq -2A(y(s_1) - y(s_0))$$

As y' vanishes at s_1, $V(s_1) = B$ and $V(s_0) = A(y'(s_0))^2 + B$,

$$\ln \frac{V(s_1)}{V(s_0)} \geq -2AM$$

$$\ln \frac{V(s_1)}{V(s_0)} \leq 4AM.$$

$$\frac{V(s_1)}{V(s_0)} \leq e^{4AM}$$

$$A(y'(s_0))^2 + B \leq Be^{4AM}$$

$$|y'(s_0)| \leq \sqrt{\frac{B}{A}(e^{4AM} - 1)} := M_1$$

This holds for any $s_0 < s_1$ in the interval.

So, we have an upper bound on $|y'(s)|$ for all $s \in [0,1]$. Thus, on the compact set $[0,1] \times [-M, M] \times [-M_1, M_1]$,

$$|f(s,u,v)| \leq M_2$$

Therefore

$$|y''(s)| = |f_\lambda(s, y(s), y'(s))| \leq |f(s, y(s), y'(s)| < M_2,$$

and

$$\|y\|_2 = \|y\| + \|y'\| + \|y''\| < \hat{M} = M + M_1 + M_2 + 1.$$

By the Leray-Schauder Alternative, $S : C_0^2 \to C_0^2$ has a fixed point $Sy = y$ (as $Sy \neq \lambda y$ for all $\lambda > 1$ if $\|y\|_2 = \hat{M}$. Λ

Example: Consider the following example:

$$y'' = f(y, y, y'),\ y(0) = y(1) = 0$$

where

$$f(t,u,v) = p(t)u^\gamma + q(t)v^\alpha + r(t)$$

where p, q, r are continuous on $[0,1]$, $0\alpha \leq 2$, and $\gamma > 0$. We suppose $\exists M > 0$ such that $|u| > M$ implies $u(p(t)u^\gamma + r(t)) > 0$ (this is a condition on $p(t)$, $r(t)$, γ) so condition 1 holds. Then, it follows that

$$|f(t,u,v)| \leq Av^2 + B$$

will hold for some A, B if $\|u\| \leq M$, $0 \leq t \leq 1$ and all v (by continuity).

6.7 SUMMARY

- We study in this unit Gronwall's Inequality

  Let u, v be nonnegative continuous functions $[a, b]$ such that

$$v(t)C + \int_a^t v(s)u(s)ds, \; a \leq t \leq b,$$

  then

$$v(t) \leq Ce^{\int_a^t u(s)ds}$$

  In particular, if $C = 0$, then $v = 0$.
- We study the theorems in the BVP

$$y'' = f(t, y, y')$$

$$y(0) = y(1) = 0$$

  where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous

- We study a rod of finite length whose ends are kept at a fixed temperature. Under certain simplifying asumptions, the temperature at any point $s \in [0,1]$ satisfies a certain second order differential equation of the form

$$ky'' + k'(y')^2 + q(s, y) = 0, \; 0 < s < 1,$$

6.8 KEYWORD

Inequality : the relation between two expressions that are not equal, employing a sign such as ≠ ‘not equal to’, > ‘greater than’, or < ‘less than

Parametrized family : a parametric family or a parameterized family is a family of objects (a set of related objects) whose differences depend only on the chosen values for a set of parameters. Common examples are parametrized (families of) functions, probability distributions, curves, shapes, etc

Integral Invariant : An absolute integral invariant is an exterior differential form of degree that is transformed into itself by the transformations generated by this system

6.9 EXERCISE

Q. 1 *Let* $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ *be continuous and satisfy the growth conditions stated previously. Then there exists a* $r>0$ *such that if* $y=y(s)$ *is a solution of*

$$y''=f_\lambda(s,y,y')$$

$$y(0)=y(1)=0$$

for $\lambda>1$*, then* $\|y\|_2<r$.

Q. 2 If $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is bounded, then the BVP

$$y''=f(t,y,y')$$

$$y(0)=y(1)=0$$

has a solution.

Q. 3 If X is a complete and totally bounded metric space, then X is compact.

Q. 4 State and Prove Gronwall's Inequality.

Q. 5 Define Continuation Differential Equation.

Q. 6 The operator $S:C_0^2[0,1]\to C_0^2[0,1]$,

given by, $(S(y))(t)=L^{-1}(F(j(y))(t)$

is completely continuous.

6.10 ANSWER TO CHECK IN PROGRESS

Check In Progress-1

Answer Q. 1 Check in section 3

Q. 2 Check in section 4

Check In progress-II

Answer Q. 1 Check in Section 6.2

Q. 2 Check in Section Lemma 7.1

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UNIT 7 MAXIMAL INTERVAL OF EXISTENCE AND CONTINUOUS DEPENDENCE

STRUCTURE

7.0 OBJECTIVES

- We study in this unit Maximal interval of Existence.
- We also study Continuous analogues of iteration method
- We study adjustment method with examples
- We study differential equation, partial, discontinuous initial (boundary) conditions

7.1 INTRODUCTION

Notes

Definition. (Maximal interval of existence) The interval (α, β) in Theorem 1 is called the maximal interval of existence of the solution x(t) of the initial value problem (1) or simply the maximal interval of existence of the initial value problem (1). x(t) = L) , then L ∈ ˙ E. In this section we give some sufficient conditions under which every local solution of an IVP is global. One of them is the growth of f wrt y. If the growth is at most linear, then we have a global solution.

In situations where a physical process is described (modelled) by an initial value problem for a system of ODEs, then it is desirable that any errors made in the measurement of either initial data or the vector field, do not influence the solution very much. In mathematical terms, this is known as continuous dependence of solution of an IVP, on the data present in the problem. In fact, the following result asserts that solution to an IVP has not only continuous dependence on initial data but also on the vector field f.

7.2 CONTINUOUS ANALOGUES OF ITERATION METHODS

Continuous models that make it possible to study problems concerning the existence of solutions of non-linear equations, to produce by means of the well-developed apparatus of continuous analysis preliminary results on the convergence and optimality of iteration methods, and to obtain new classes of such methods.

One can set up a correspondence between methods for solving stationary problems by adjustment (see Adjustment method) and certain iteration methods. For example, for the solution of a linear equation

$$Au = f$$

with a positive-definite self-adjoint operator A it is known that one-step iteration methods of the form

$$\frac{(u^{k+1} - u^k)}{\tau_k} = -(Au^k - f), \qquad u^0 = v,$$

converge for sufficiently small $\tau_k > 0$. Introduce a continuous time t and regard the quantities u^k as the values of a certain function $u(t)$ at $t = t_k$, where

$$t_0 = 0 < t_1 < \ldots < t_k < \ldots, \qquad t_k \to \infty \ \text{ as } k \to \infty \ .$$

If one puts $\tau_k = \rho(t_k)(t_k - t_{k-1})$, where $\rho(t) > 0$ is a continuous function for $t \geq 0$, then in passing to the limit in (2) as $\Delta t_k = t_{k+1} - t_k \to 0$, one obtains a continuous analogue of the iteration method (2):

$$\frac{du}{dt} = -\rho(t)(Au - f), \qquad u(0) = v \,.$$

If also

$$\int_0^t \rho(t)\,dt \to \infty$$

as $t \to \infty$, then $u(t)$ tends to $u(\infty)$, a solution of (1).

Similarly, with the one-step gradient iteration methods for the minimization of a function $F(u)$:

$$u^{k+1} = u^k - \tau_k \text{ grad } F(u^k), \qquad u^0 = v,$$

one can associate a continuous analogue:

$$\frac{du}{dt} = -\rho(t) \text{ grad } F(u), \qquad u(0) = v \,.$$

Here the function $\rho(t)$ affects only the parametrization of the curve of steepest descent. To solve (1) one may take $F(u) = (Au, u) - 2(f, u)$. Then the formulas (4) take the form (2) and the equations (5) the form (3).

By means of transformations two-step iteration methods

$$u^{k+1} = u^k - \alpha_k (Au^k - f) - \beta_k (u^k - u^{k-1})$$

can be brought to the form

$$2\frac{\left[\frac{u^{k+1} - u^k}{\Delta t_k} - \frac{u^k - u^{k-1}}{\Delta t_{k-1}}\right]}{t_{k+1} - t_{k-1}} +$$

$$+ \gamma_k \frac{u^{k+1} - u^{k-1}}{t_{k+1} - t_{k-1}} + \mu_k u^k = -\rho_k (Au^k - f),$$

Notes

where the quantities γ_k, μ_k, ρ_k, and t_k are (non-uniquely) defined in terms of the parameters α_k and β_k of (6). Taking limits in (7) as $\Delta t_k \to 0$ leads to a continuous analogue:

$$\frac{d^2u}{dt^2} + \gamma(t)\frac{du}{dt} + \mu(t)u = -\rho(t)(Au - f).$$

The adjustment method involving an equation like (8) is called the method of the heavy sphere . There exist iteration methods for which the continuous analogues contain differential operators of higher orders .

A source for obtaining differential equations playing the role of continuous analogues of iteration methods can be the continuation method (with respect to a parameter). In this method, to find a solution of an equation

$$\phi(u) = 0$$

one constructs an equation

$$\Phi(u, \lambda) = 0,$$

depending on a parameter λ, such that for $\lambda = 0$ a solution of (10) is known: $u(0) = u^0$, and such that for $\lambda = 1$ the solutions of (9) and (10) are the same. For example, one can take

$$\Phi(u, \lambda) = \phi(u) - (1 - \lambda)\phi(u^0).$$

By differentiating (1) with respect to the parameter and taking $u = u(\lambda)$ one obtains a differential equation for $u(\lambda)$; for the case (11) it takes the form

$$\frac{du}{d\lambda} = -\phi'(u)^{-1}\phi(u^0).$$

By splitting the interval $[0, 1]$ into n parts by points $\lambda_0 = 0 < \lambda_1 < \ldots < \lambda_n = 1$ and using for (12) a numerical discretization formula at the points λ_k (e.g., Euler's method, the Runge–Kutta method, etc.), one obtains recurrence relations between the quantities $u^k = u(\lambda_k)$, which one uses to construct the formulas of an iteration method. Thus, after e.g. applying Euler's method, (12) is replaced by the relations

$$u^k = u^{k-1} - \Delta \lambda_k \phi' (u^{k-1})^{-1} \phi (u^0),$$

where $\Delta \lambda_k = \lambda_k - \lambda_{k-1}$, which determine the following two-step iteration method containing internal and external iteration cycles:

$$u_k^i = u_{k-1}^i - \Delta \lambda_k \phi' (u_{k-1}^i)^{-1} \phi (u_0^i),$$
$$k = 1, \dots, n; \qquad u_0^i = y_n^{i-1}, \qquad i = 1, 2, \dots, \qquad u_0^0 = u^0 .$$

For $\Delta \lambda_1 = 1$ and $n = 1$ this turns into Newton's classical method. A continuous analogue of Newton's iteration method can also be obtained in another way: In (11) the variable is replaced by $\lambda = 1 - e^{-t}$. Then the differential equation (12) takes the form

$$\frac{du}{dt} = \phi' (u)^{-1} \phi (u), \qquad u(0) = u^0 .$$

Numerical integration of (15) by Euler's method with respect to the points t_k leads to the iteration method

$$u^k = u^{k-1} - \Delta t^{k-1} \phi' (u^{k-1})^{-1} \phi (u^{k-1}),$$

which coincides for $\Delta t_{k-1} = 1$ with Newton's classical method.

Continuous analogues of iteration methods for the solution of boundary value problems for the differential equations of mathematical physics are, as a rule, mixed problems for partial differential equations of a special form (e.g. with rapidly oscillating coefficients or with small coefficients in front of the highest derivatives).

7.3 ADJUSTMENT METHOD

A method in which the solution u of a stationary problem

$$Au = f$$

is regarded as the steady-state limit solution for $t \to \infty$ of a Cauchy initial value problem for a non-stationary evolution equation involving the same operator A (cf. Cauchy problem). This evolution equation may e.g. be of the form

$$\sum_{i=1}^{m} C_i \frac{d^i u(t)}{dt^i} = f - Au(t),$$

$$\left.\frac{d^k u}{dt^k}\right|_{t=0} = u_{0k}, \qquad k = 0, \ldots, m-1 .$$

Here the C_i are suitable operators which guarantee the existence of the "adjustment limit" $\lim_{t \to \infty} u(t) = u$.

A result of using adjustment is that it permits one to use approximate solution methods of (2) in order to construct iteration algorithms for solving equation (1) (cf. Iteration algorithm). Thus, for the non-stationary equation (2) one could employ a discretization (differencing) with respect to t solution method which is convergent and stable to obtain approximate solutions. For example, for $m = 1$, an explicit method of the form

$$C_1 \frac{u(t_{n+1}) - u(t_n)}{\tau_n} = f - Au(t_n)$$

where $\tau_n = t_{n+3} - t_n > 0$. And then this method can be interpreted as an iteration algorithm

$$C_1 (u^{n+1} - u^n) = \tau_n \, (f - Au^n), \qquad n = 0, 1, \ldots, \qquad u^0 = u_{00},$$

for solving equation (1), in which C_1 and τ_n are now seen as characterizing this (iteration) method.

Varying the form of the operators C_i and considering different discretizations with respect to t in equation (2) (explicit schemes, implicit schemes, splitting schemes, etc.) gives the possibility of obtaining a wide variety of iteration methods for solving equation (1). For these methods equation (2) will be the closure of the computational algorithm (cf. Closure of a computational algorithm). A generalization of the adjustment method is the continuation method (to a parametrized family)

7.3.1 Intervals Of Validity

We've called this section Intervals of Validity because all of the examples will involve them. However, there is a lot more to this section. We will see a couple of theorems that will tell us when we can solve a differential equation. We will also see some of the differences between linear and nonlinear differential equations.

First let's take a look at a theorem about linear first order differential equations. This is a very important theorem although we're not going to really use it for its most important aspect.

Theorem 1 Consider the following IVP.

$y' + p(t)y = g(t) y(t0) = y0$

If p(t) and g(t) are continuous functions on an open interval $\alpha < t < \beta$ and the interval contains to, then there is a unique solution to the IVP on that interval.

So, just what does this theorem tell us? First, it tells us that for nice enough linear first order differential equations solutions are guaranteed to exist and more importantly the solution will be unique. We may not be able to find the solution but do know that it exists and that there will only be one of them. This is the very important aspect of this theorem. Knowing that a differential equation has a unique solution is sometimes more important than actually having the solution itself!

Next, if the interval in the theorem is the largest possible interval on which p(t) and g(t) are continuous then the interval is the interval of validity for the solution. This means, that for linear first order differential equations, we won't need to actually solve the differential equation in order to find the interval of validity. Notice as well that the interval of validity will depend only partially on the initial condition. The interval must contain to, but the value of yo, has no effect on the interval of validity.

Let's take a look at an example.

Example 1 Without solving, determine the interval of validity for the following initial value problem.$(t^2-9)\, y' + 2y = \ln|20-4t|$

$y(4) = -3$

Solution: First, in order to use the theorem to find the interval of validity we must write the differential equation in the proper form given in the theorem. So we will need to divide out by the coefficient of the derivative.

$y' + 2/\ t^2-9\ y = \frac{\ln|20-4t|}{t^2-9}\ y'+2t$

Next, we need to identify where the two functions are not continuous. This will allow us to find all possible intervals of validity for the differential equation. So, p(t) will be discontinuous at $t=\pm3$ since these points will give a division by zero. Likewise, g(t) will also be discontinuous at $t=\pm3$ as well as t=5t since at this point we will have the natural logarithm of zero. Note that in this case we won't have to worry about natural log of negative numbers because of the absolute values.

Now, with these points in hand we can break up the real number line into four intervals where both p(t)p(t) and g(t)g(t) will be continuous. These four intervals are,

$-\infty<t<-3$ $-3<t<3$ $3<t<5$ $5<t<\infty$

The endpoints of each of the intervals are points where at least one of the two functions is discontinuous. This will guarantee that both functions are continuous everywhere in each interval.

Finally, let's identify the actual interval of validity for the initial value problem. The actual interval of validity is the interval that will contain to=4 . So, the interval of validity for the initial value problem is.

$$3<t<5$$

7.4 DIFFERENTIAL EQUATION, PARTIAL, DISCONTINUOUS INITIAL (BOUNDARY) CONDITIONS

A problem involving partial differential equations in which the functions specifying the initial (boundary) conditions are not continuous.

For instance, consider the second-order hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f, \qquad 0 \langle x < 1, \qquad t \rangle t_0,$$

and pose for it the mixed problem with initial conditions

$$\left.\frac{\partial u}{\partial t}\right|_{t=t_0} = \phi_1 , \qquad u|_{t=t_0} = \phi_0 ,$$

and boundary conditions

$$u|_{x=0} = \psi_1 , \qquad u|_{x=1} = \psi_2 \ .$$

In this case the discontinuities of the initial functions ϕ_0 and ϕ_1 entail discontinuities of u and $\partial u / \partial t$ along the characteristic rays $x - at = \text{const}$ and $x + at = \text{const}$, and the measure of discontinuity

$$\chi = u(c \pm at + 0, t) - u(c \pm at - 0, t),$$

or

$$\chi = u_t(c \pm at + 0, t) - u_t(c \pm at - 0, t),$$

where $c \in [0, 1]$ is a discontinuity point of the function ϕ_0 or ϕ_1, satisfies the equation

$$\frac{d\chi}{dt} + 0.\chi = 0$$

along the characteristic ray, i.e. $\chi = \text{const}$. Similar results are valid for second-order hyperbolic equations with variable coefficients:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u + f,$$

$$u|_{t=t_0} = \phi_0 (x), \qquad u_t|_{t=t_0} = \phi_1 (x), \qquad u|_{\partial D} = \psi \ .$$

In this case the discontinuities of the initial functions and the boundary conditions also entail discontinuities in u and $\partial u / \partial t$ along characteristic rays, which can be determined from the systems of equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(x) \phi_j , \qquad \sum_{i,j=1}^{n} a_{ij}(x) \phi_i \ \phi_j = 0 .$$

Notes

The measure of discontinuity χ satisfies the equation:

$$2\frac{d\chi}{dt}+A\chi=0, \qquad A=\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2\phi}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i\frac{\partial\phi}{\partial x_i},$$

where the function $\phi(x)$ defines the characteristic surface in the form of the equation $\phi(x)=C$.

In the case of equations of elliptic type the discontinuities of the boundary conditions do not propagate inside D because in this case the characteristic rays are complex. For equations of elliptic type studies were made of the existence and uniqueness of the solution, and of the solution satisfying the boundary conditions. Thus, it has been proved for second-order elliptic equations in an arbitrary domain,

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}+c(x)u=f,$$

$$u|_{\partial D}=\psi \quad \text{or} \quad \frac{\partial u}{\partial n}+k(x)u|_{\partial D}=\psi,$$

that if the boundary function $\psi\in W_2^{1/2}(\partial D)$ for the first boundary condition and $\psi\in L_2(\partial D)$ for the second boundary condition, then there exists a generalized solution in $W_2^1(D)$ which satisfies the boundary condition on the average, i.e. $\|u-\psi\|_{L_2(\partial D_n)}\to 0$, where the surfaces ∂D_n approximate the surface ∂D. In the case of parabolic (and also elliptic) equations, the discontinuities do not propagate inside D if discontinuities are present in the initial or in the boundary conditions. Problems of the existence and uniqueness of a generalized solution to the boundary condition have also been studied for these problems.

Check In Progress-I

Q. 1 Write second order hyperbolic differential equation.

Solution : . . . .

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. . . .

Q. 2 Define Adjusment Method.

Solution : . . . .

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. . . .

7.5 HOMOGENEOUS EQUATIONS

The differential equation

$$(H) \qquad \frac{dy}{dx} = f(x,y),$$

is *homogeneous* if the function $f(x,y)$ is homogeneous, that is-

$$f(tx,ty) = f(x,y) \text{ for any number } t.$$

Check that the functions

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{and} \quad f(x,y) = \ln\left(\frac{-3x^2y}{x^3+4xy^2}\right).$$

are homogeneous.

In order to solve this type of equation we make use of a substitution (as we did in case of Bernoulli equations). Indeed, consider the substitution $z = \frac{y}{x}$. If $f(x,y)$ is homogeneous, then we have

$$f(x,y) = f(1,z) = F(z).$$

Since $y' = xz' + z$, the equation (H) becomes

$$x\frac{dz}{dx} + z = f(1, z)$$

which is a separable equation. Once solved, go back to the old variable *y* via the equation *y* = *x z*.

Let us summarize the steps to follow:

(1) Recognize that your equation is an homogeneous equation; that is, you need to check that *f*(*tx*,*ty*)= *f*(*x*,*y*), meaning that *f*(*tx*,*ty*) is independent of the variable *t*;

(2) Write out the substitution *z*=*y*/*x*;

(3) Through easy differentiation, find the new equation satisfied by the new function *z*.

You may want to remember the form of the new equation:

$$x\frac{dz}{dx} + z = f(1, z)\ .$$

(4) Solve the new equation (which is always separable) to find *z*;

(5) Go back to the old function *y* through the substitution *y* = *x z*;

(6) If you have an IVP, use the initial condition to find the particular solution.

Since you have to solve a separable equation, you must be particularly careful about the constant solutions.

Example: Find all the solutions of

$$\frac{dy}{dx} = \frac{-2x + 5y}{2x + y}\ .$$

Solution: Follow these steps:

(1) It is easy to check that $f(x, y) = \dfrac{-2x + 5y}{2x + y}$ is homogeneous;

(2) Consider $z = \dfrac{y}{x}$;

(3) We have

$$xz' + z = \frac{-2x + 5xz}{2x + xz} = \frac{-2 + 5z}{2 + z},$$

which can be rewritten as

$$z' = \frac{1}{x}\left(\frac{-2 + 5z}{2 + z} - z\right).$$

This is a separable equation. If you don't get a separable equation at this point, then your equation is not homogeneous, or something went wrong along the way.

(4) All solutions are given implicitly by

$$\begin{cases} -4\ln(|z-2|) + 3\ln(|z-1|) = \ln(|x|) + C \\ z = 1 \\ z = 2. \end{cases}$$

(5) Back to the function *y*, we get

$$\begin{cases} -4\ln(|y-2x|) + 3\ln(|y-x|) = C \\ y = x \\ y = 2x. \end{cases}$$

Note that the implicit equation can be rewritten as

$$(y - x)^3 = C_1(y - x)^4 .$$

7.6 SEPARABLE EQUATIONS

The differential equation of the form $\frac{dy}{dx} = f(x,y)$ is called **separable**, if $f(x,y) = h(x)\,g(y)$; that is,

$$(S) \qquad \frac{dy}{dx} = h(x)g(y).$$

In order to solve it, perform the following steps:

(1) Solve the equation $g(y) = 0$, which gives the constant solutions of (S);

(2) Rewrite the equation (S) as

$$\frac{dy}{g(y)} = h(x)dx,$$

and, then, integrate

$$\int \frac{1}{g(y)}dy = \int h(x)dx.$$

to obtain

$$G(y) = H(x) + C.$$

(3) Write down all the solutions; the constant ones obtained from (1) and the ones given in (2);

(4) If you are given an IVP, use the initial condition to find the particular solution. Note that it may happen that the particular solution is one of the constant solutions given in (1). This is why Step 3 is important.

Example: Find the particular solution of

$$\frac{dy}{dx} = \frac{y^2 - 1}{x} \quad y(1) = 2.$$

Solution: Perform the following steps:

(1) In order to find the constant solutions, solve $y^2 - 1 = 0$. We obtain $y = 1$ and $y=-1$.

(2) Rewrite the equation as

$$\frac{dy}{y^2 - 1} = \frac{dx}{x}.$$

Using the techniques of integration of rational functions, we get

$$\int \frac{dy}{y^2 - 1} = \frac{1}{2}\ln\left(\frac{|y - 1|}{|y + 1|}\right),$$

which implies

$$\frac{1}{2}\ln\left(\frac{|y-1|}{|y+1|}\right)=\ln(|x|)+C\ .$$

(3) The solutions to the given differential equation are

$$\begin{cases}\frac{1}{2}\ln\left(\frac{|y-1|}{|y+1|}\right)=\ln(|x|)+C\\ y=1\\ y=-1.\end{cases}$$

(4) Since the constant solutions do not satisfy the initial condition, we are left to find the particular solution among the ones found in (2), that is we need to find the constant *C*. If we plug in the condition *y*=2 when *x*=1, we get

$$\frac{1}{2}\ln\left(\frac{1}{3}\right)=C\ .$$

Note that this solution is given in an implicit form. You may be asked to rewrite it in an explicit one. For example, in this case, we have

$$y=\frac{3+x^2}{3-x^2}\ .$$

Example: Find all solutions to

$$\frac{dy}{dt}=1+\frac{1}{y^2}.$$

Solution: First, we look for the constant solutions, that is, we look for the roots of

$$1+\frac{1}{y^2}=0\ .$$

This equation does not have real roots. Therefore, we do not have constant solutions.

Notes

The next step will be to look for the non-constant solutions. We proceed by separating the two variables to get

$$\frac{dy}{1+\frac{1}{y^2}} = dt$$
.

Then we integrate

$$\int \frac{dy}{1+\frac{1}{y^2}} = \int dt \ .$$

Since

$$\frac{1}{1+\frac{1}{y^2}} = \frac{y^2}{y^2+1} = 1 - \frac{1}{y^2+1}$$

we get

$$\int \frac{dy}{1+\frac{1}{y^2}} = y - \tan^{-1}(y) \ .$$

Therefore, we have

$$y - \tan^{-1}(y) = t + C \ .$$

It is not easy to obtain *y* as a function of *t*, meaning finding *y* in an explicit form.

Finally, because there are no constant solutions, all the solutions are given by the implicit equation

$$y - \tan^{-1}(y) = t + C \ .$$

Example: Solve the initial value problem

$$\frac{dy}{dt} = 1 + t^2 + y^2 + t^2y^2 \qquad y(0) = 1.$$

Answer: This is a separable equation. Indeed, we have

$$1 + t^2 + y^2 + t^2y^2 = (1 + t^2)(1 + y^2) \ .$$

Before we get into integration we need to look for the constant solutions. These are the roots of the equation $1 + y^2 = 0$. Since this equation has no real roots, we conclude that no-constant solution exists. Therefore, we proceed with the separation of the two variables and integration. We have

$$\frac{dy}{1 + y^2} = (1 + t^2)dt \ ,$$

which gives

$$\int \frac{dy}{1 + y^2} = \int (1 + t^2)dt \ .$$

Since

$$\int \frac{dy}{1 + y^2} = \tan^{-1}(y)$$

and

$$\int (1 + t^2)dt = t + \frac{t^3}{3} ,$$

we get

$$\tan^{-1}(y) = t + \frac{t^3}{3} + C \ .$$

The initial condition $y(0)=1$ gives

$$C = \tan^{-1}(1) = \frac{\pi}{4} \ .$$

The particular solution to the initial value problem is

$$\tan^{-1}(y) = t + \frac{t^3}{3} + \frac{\pi}{4},$$

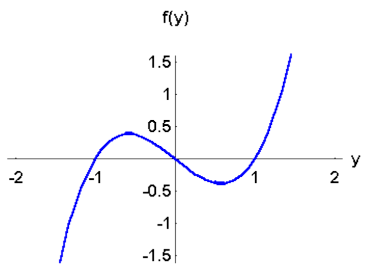
or in the explicit form

$$y = \tan\left(t + \frac{t^3}{3} + \frac{\pi}{4}\right).$$

Example: Consider the autonomous differential equation

$$\frac{dy}{dt} = f(y),$$

where the graph of *f*(*y*) is given by



1.Sketch the Slope Fields of this differential equation.

(Hint: the graph of the solutions and the graph of *f*(*y*) are two different entities.)

2.Sketch the graph of the solution to the IVP

$$\frac{dy}{dt} = f(y), \quad y(0) = \frac{1}{2}.$$

Find the limit $\lim_{t\to+\infty} y(t)$.

3.Sketch the graph of the solution to the IVP

$$\frac{dy}{dt} = f(y), \quad y(0) = -\frac{1}{2}.$$

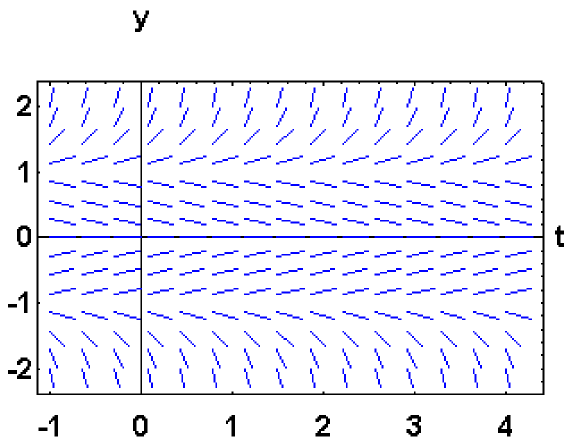
Find the limit $\lim_{t\to+\infty} y(t)$.

Answer: 1. Since we do not know the function $f(y)$, we will only be able to sketch the slope fields. This will give us an idea about the behavior of the solutions. Therefore, we should be looking for the critical solutions (given by the roots of $f(y)=0$), and the sign of $f(y)$ which will give the variation of the solutions. Note that we should be careful not to mix between the graph of $f(y)$ and the graphs of the solutions $y(t)$.

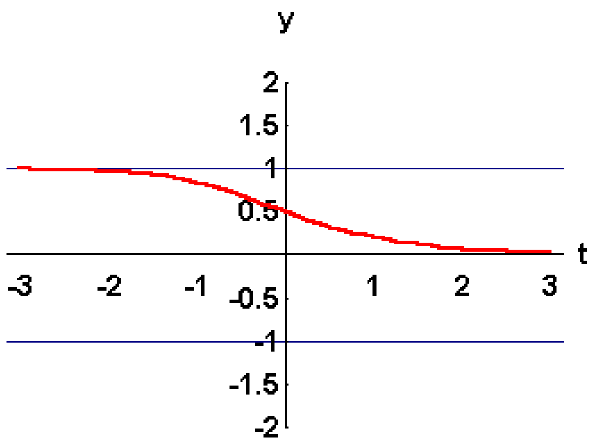
So, according to the graph of $f(y)$, the critical solutions are $y= -1$, $y=0$, and $y=1$. Using the sign of $f(y)$, we conclude that

- the solutions located in the region $y<-1$ are decreasing,
- the solutions located in the region $-1 < y<0$ are increasing,
- the solutions located in the region $0<y<1$ are decreasing,
- the solutions located in the region $1 < y$ are increasing.

The sketch of the slope fields is given below.



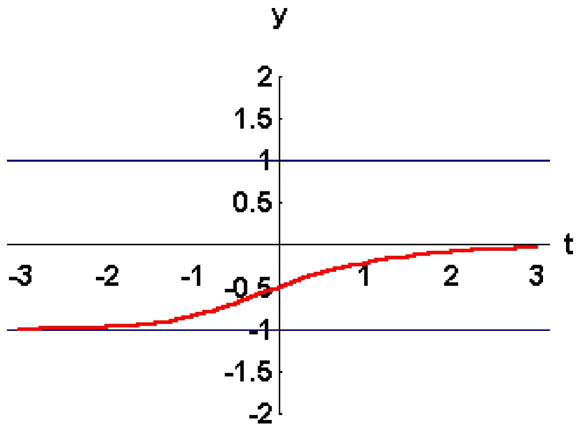
2.Using the slope fields, we sketch the graph of the solution satisfying the initial condition $y(0) = 0.5$.



Clearly, we have

$$\lim_{t \to +\infty} y(t) = 0 \; .$$

3.Using the slope fields, we sketch the graph of the solution satisfying the initial condition $y(0) = -0.5$.



Clearly, we have

$$\lim_{t \to +\infty} y(t) = 0 \; .$$

Check In Progress-II

Q. 1 Define Homogeneous Equation.

Solution : . . . .

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Q. 2 Find one solutions to the given differential equation :

$$\frac{dy}{dt} = 1 + \frac{1}{y^2}.$$

Solution : . . . .

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. . . .

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. . . .

. . . .

7.7 HERMITE'S EQUATION

Hermite's Equation of order *k* has the form

$$y''-2ty'+2ky=0,$$

where *k* is usually a non-negative integer.

We know from the previous section that this equation will have series solutions which both converge and solve the differential equation everywhere.

Hermite's Equation is our first example of a differential equation, which has a polynomial solution.

As usual, the generic form of a power series is

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

We have to determine the right choice for the coefficients (a_n).

As in other techniques for solving differential equations, once we have a "guess" for the solutions, we plug it into the differential equation. Recall that

$$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1},$$

and

$$y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}.$$

Plugging this information into the differential equation we obtain:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2t\sum_{n=1}^{\infty} na_n t^{n-1} + 2k\sum_{n=0}^{\infty} a_n t^n = 0,$$

or after rewriting slightly:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=1}^{\infty} 2na_n t^n + \sum_{n=0}^{\infty} 2ka_n t^n = 0.$$

Next we shift the first summation up by two units:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=1}^{\infty} 2na_n t^n + \sum_{n=0}^{\infty} 2ka_n t^n = 0.$$

Before we can combine the terms into one sum, we have to overcome another slight obstacle: the second summation starts at n=1, while the other two start at n=0.

Evaluate the 0th term for the second sum: $2 \cdot 0 \cdot a_0 \cdot t^0 = 0$.

Consequently, we do not change the value of the second summation, if we start at n=0 instead of n=1:

$$\sum_{n=1}^{\infty} 2na_n t^n = \sum_{n=0}^{\infty} 2na_n t^n.$$

Thus we can combine all three sums as follows:

$$\sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} - 2na_n + 2ka_n \right) t^n = 0.$$

Therefore our recurrence relations become:

$$(n+2)(n+1)a_{n+2} - 2na_n + 2ka_n = 0 \text{ for all } n = 0, 1, 2, 3, \ldots$$

After simplification, this becomes

$$a_{n+2} = \frac{2(n-k)}{(n+2)(n+1)} a_n = 0 \text{ for all } n = 0, 1, 2, 3, \ldots$$

Let us look at the special case, where k=5, and the initial conditions are given as $y(0) = a_0 = 0, \; y'(0) = a_1 = 1$. In this case, all even coefficients will be equal to zero, since a_0=0 and each coefficient is a multiple of its second predecessor.

$$a_0 = a_2 = a_4 = a_6 = \ldots = 0.$$

What about the odd coefficients? a_1=1, consequently

$$a_3 = \frac{2(1-5)}{2 \cdot 3} = -\frac{4}{3},$$

and

$$a_5 = \frac{2(3-5)}{4 \cdot 5} a_3 = (-\frac{1}{5})(-\frac{4}{3}) = \frac{4}{15}.$$

What about a_7:

$$a_7 = \frac{2(5-5)}{6 \cdot 7} a_5 = 0.$$

Since a_7=0, all odd coefficients from now on will be equal to zero, since each coefficient is a multiple of its second predecessor.

$$a_7 = a_9 = a_{11} = a_{13} = \ldots = 0.$$

Consequently, the solution has only 3 non-zero coefficients, and hence is a polynomial. This polynomial

$$H_5(t) = t - \frac{4}{3}t^3 + \frac{4}{15}t^5$$

(or a multiple of this polynomial) is called the **Hermite Polynomial of order 5**.

Notes

It turns out that the Hermite Equation of positive integer order *k* always has a polynomial solution of order *k*. We can even be more precise:

If *k* is odd, the initial value problem $a_0 = 0,\ a_1 = 1$ will have a polynomial solution, while for *k* even, the initial value problem $a_0 = 1,\ a_1 = 0$ will have a polynomial solution.

Exercise 1: Find the Hermite Polynomials of order 1 and 3.

Recall that the recurrence relations are given by

$$a_{n+2} = \frac{2(n-k)}{(n+2)(n+1)} a_n = 0 \text{ for all } n = 0, 1, 2, 3, \ldots$$

We have to evaluate these coefficients for k=1 and k=3, with initial conditions a_0=0, a_1=1.

When k=1,

$$a_3 = \frac{2(1-1)}{2 \cdot 3} a_1 = 0.$$

Consequently all odd coefficients other than a_1 will be zero. Since a_0=0, all even coefficients will be zero, too. Thus

$$H_1(t) = t.$$

When k=3,

$$a_3 = \frac{2(1-3)}{2 \cdot 3} a_1 = -\frac{2}{3},$$

and

$$a_5 = \frac{2(3-3)}{4 \cdot 5} a_3 = 0.$$

Consequently all odd coefficients other than a_1 and a_3 will be zero.

Since a_0=0, all even coefficients will be zero, too. Thus

$$H_3(t) = t - \frac{2}{3}t^3.$$

7.8 SUMMARY

- We study for the solution of a linear equation

  $Au = f$

  with a positive-definite self-adjoint operator A it is known that one-step iteration methods of the form

$$\frac{(u^{k+1} - u^k)}{\tau_k} = -(Au^k - f), \qquad u^0 = v,$$

- We leant A method in which the solution u of a stationary problem

  $Au = f$

  is regarded as the steady-state limit solution for $t \longrightarrow \infty$ of a Cauchy initial value problem for a non-stationary evolution equation involving the same operator A

- Also learnt **Hermite's Equation** of order k has the form

$$y''-2ty'+2ky=0,$$

where k is usually a non-negative integer.

- We learnt The differential equation of the form $\frac{dy}{dx} = f(x,y)$ is called **separable**, if $f(x,y) = h(x)\,g(y)$; that is,

$$(S) \qquad \frac{dy}{dx} = h(x)g(y).$$

7.9 KEYWORD

Hermite Equation : The *Hermite polynomials* are set of orthogonal *polynomials* over the domain with weighting function , illustrated above for , 2, 3, and 4. *Hermite polynomials* are implemented in the Wolfram Language as HermiteH[n, x]. The *Hermite polynomial* can be *defined* by the contour integral.

Seperable Function : Separable Functions. means that f(x) is a constant. For a function of two variables G y(x, y) = 0 means that. G(x, y) is a constant function of y but can be an arbitrary function of x.

Iteration : Sepetition of a mathematical or computational procedure applied to the result of a previous application, typically as a means of obtaining successively closer approximations to the solution of a problem.

7.10 EXCERCISE

Q. 1 Consider the Hermite Equation of order 5:

$$y''-2ty'+10y=0.$$

Find the solution satisfying the initial conditions $a_0=1$, $a_1=0$.

Q. 2 Find the Hermite Polynomials of order 2, 4 and 6.

Q. 3 Define Hermite's Equation .

Q. 4 Find all solutions to

$$\frac{dy}{dt} = 1 + \frac{1}{y^2}.$$

Q. 5 Find all the solutions of

$$\frac{dy}{dx} = \frac{-2x+5y}{2x+y} .$$

7.11 ANSWER TO CHECK IN PROGRESS

Check In Progress-I

Answer Q. 1 Check in Section 4

Q. 2 Check in Section 5

Check In progress-II

Answer Q. 1 Check in Section 6

Q. 2 Check in Section 7

7.12 SUGGESTION READING AND REFERENCES

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